Algebraic Topology

Cambridge University Mathematical Tripos: Part II

4th May 2024

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1 Motivation

1.1 Invariants

Topological spaces are difficult to study on their own, and so we will assign algebraic invariants to these spaces which allow us to reason more easily about these spaces. To a topological space X, a 'numerical invariant' is a number $g(X) \in \mathbb{R} \cup \{\infty\}$ such that $X \simeq Y$ (where $\simeq$ denotes homeomorphism) implies $g(X) = g(Y)$. An example of a numerical invariant is the number of path-connected components of X. An algebraic invariant is a group $G(X)$ assigned to a topological space X such that $X \simeq Y$ implies $G(X) \simeq G(Y)$, where here $\simeq$ denotes isomorphism. We will construct two kinds of such invariants: the fundamental group, and invariants related to homology. The invariants we construct will behave nicely under maps: if $f : X \to Y$ is a continuous map, we induce a homomorphism $f_\star : G(X) \to G(Y)$. We will prove the following model theorems.

- If $\mathbb{R}^n \simeq \mathbb{R}^m$, then $n = m$.
- If $f : D^n \to D^n$ is continuous, then there exists $x \in D^n$ with $f(x) = x$.

The above theorems are easy to prove in the case $n = 1$ by appealing to path-connectedness and the intermediate value theorem. Our study allows us to prove similar things about these higher dimensional cases, among other things.

1.2 Notation

- A *space* is a topological space.
- A *map* is a continuous function, unless defined otherwise.
- If X and Y are spaces, $X \simeq Y$ means that X and Y are homeomorphic.
- If G and H are groups, $G \simeq H$ means that G and H are isomorphic.
- Some common spaces include:
  - The one-point space $\{\bullet\}$;
  - $I = [0, 1] \subset \mathbb{R}$;
  - $I^n = \underbrace{I \times \dots \times I}_{n \text{ times}}$, the n-dimensional closed unit cube;
  - $D^n = \{v \in \mathbb{R}^n \mid \|v\| \leq 1\}$, the n-dimensional closed unit disk (note that $I^n \simeq D^n$);
  - $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$, the $(n-1)$-dimensional unit sphere.
- Common maps include:
  - If X is a space, the identity map $\mathrm{id}_X : X \to X$ is defined by $x \mapsto x$;
  - If X and Y are spaces with $p \in Y$, the constant map $c_{X,p} : X \to Y$ is defined by $x \mapsto p$.

2 Homotopy

2.1 Definition

Definition. Let $f_0, f_1 : X \to Y$ be continuous. We say f_0 is *homotopic to* f_1, written $f_0 \sim f_1$, if there exists a continuous $H : X \times I \to Y$ with $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$.

We can think of H as a path from f_0 to f_1 in the set $\mathrm{Hom}(X, Y)$ of functions $X \to Y$, which is continuous under a topology that will not be defined here.

Lemma (Gluing lemma)**.** Let $X = C_1 \cup C_2$, where C_1, C_2 are closed in X. Let $f : X \to Y$ be a function (that may be not continuous), such that $f|_{C_1}$ and $f|_{C_2}$ are continuous. Then f is continuous.

Proof. It suffices to show that the preimage of a closed set is closed. Let $K \subseteq Y$ be closed. Then $K_i = f^{-1}(K) \cap C_i = \left(f|_{C_i}\right)^{-1}(K)$ is a closed set in C_i and so is closed in X because C_i is closed. Since $K = K_1 \cup K_2$, K is also closed in X. □

Lemma. Homotopy is an equivalence relation.

Proof. Reflexivity is trivial, because $H(x, t) = f(x)$ is continuous, as $H = f \circ \pi_1$ is the composition of continuous maps. Symmetry holds because if $H(x, t)$ is continuous, $H(x, 1 - t)$ is continuous as the composition of continuous maps. For transitivity, if $f_0 \sim f_1$ via H and $f_1 \sim f_2$ via H', we define

$$H''(x,t) = \begin{cases} H(x, 2t) & t < \frac{1}{2} \\ H'(x, 2t-1) & t \geq \frac{1}{2} \end{cases}$$

and this is continuous by the gluing lemma. □

Note that we sometimes write $f_t(x)$ for a homotopy between f_0 and f_1.

Example. Let $f_1 : X \to \mathbb{R}^n$ be a map. Then $f_0 : X \to \mathbb{R}^n$ defined by $c_{X,0}$ has $f_1 \sim f_0$ via the homotopy $H(x, t) = t f_1(x)$.

Example. Let $f_1 : S^1 \to S^2$ be defined by $f_1(x, y) = (x, y, 0)$: the inclusion map from the circle to the equator in the unit 2-sphere. Let $f_0 : S^1 \to S^2$ be the constant map $f_0(x, y) = (0, 0, 1)$. Then $f_0 \sim f_1$ via the homotopy $f_t(x, y) = (x \sin \frac{\pi t}{2}, y \sin \frac{\pi t}{2}, \cos \frac{\pi t}{2})$.

Lemma. If $f_0, f_1 : X \to Y$ are homotopic via f_t, and $g_0, g_1 : Y \to Z$ are homotopic via g_t, then the map $H : X \times I \to Z$ defined by $H(x, t) = g_t(f_t(x))$, also denoted $g_t \circ f_t$, is a homotopy for $g_0 \circ f_0 \sim g_1 \circ f_1$.

Proof. This is a composition of continuous maps and hence continuous. □

2.2 Contractible spaces

Definition. A space Y is *contractible* if $\mathrm{id}_Y \sim c_{Y,p}$ for some $p \in Y$.

Example. If $Y \subseteq \mathbb{R}^n$ is convex and nonempty, Y is contractible via the homotopy $H(y,t) = (1-t)y + tp$ for some $p \in Y$.

Proposition. Let Y be contractible. Then $f_0 \sim f_1$ for any maps $f_0, f_1 \colon X \to Y$.

Proof. We have $f_0 = \mathrm{id}_Y \circ f_0 \sim c_{Y,p} \circ f_0 = c_{X,p}$, and similarly $f_1 \sim c_{X,p}$. By transitivity, $f_0 \sim f_1$. □

Corollary. Let Y be contractible. Then Y is path-connected.

Proof. If Y is contractible, and $p, q \in Y$, then $c_{\{\bullet\},p} \sim c_{\{\bullet\},q}$ via $H \colon \{\bullet\} \times I \to Y$. Then we can define the path $\gamma(t) = H(\bullet, t)$ from p to q in Y. □

Example. $\mathbb{R} \setminus \{0\}$ is not contractible.

We will later prove that $\mathbb{R}^n \setminus \{0\}$ is not contractible for any $n \geq 1$, but we require some more theory before this can be proven.

Definition. Spaces X, Y are *homotopy equivalent*, denoted $X \sim Y$, if there exist maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$.

Example. If $X \simeq Y$, X and Y are homotopy equivalent. Note that the definition of homotopy equivalence is simply the definition of homeomorphism, except that the requirement that $f \circ g$ and $g \circ f$ be *equal* to the identity is relaxed into simply being *homotopic* to the identity.

Lemma. Homotopy equivalence is an equivalence relation.

Proposition. X is contractible if and only if $X \sim \{\bullet\}$.

Proof. If X is contractible, $\mathrm{id} \sim c_{X,p}$. Let $f \colon X \to \{\bullet\}$ be defined by $f(x) = \bullet$. Let $g \colon \{\bullet\} \to X$ be defined by $g(x) = p$. Then $f \circ g = \mathrm{id}_{\{\bullet\}}$ and $g \circ f = c_{X,p} \sim \mathrm{id}_X$. The converse is similar. □

Example. We have $\mathbb{R}^{n+1} \setminus \{0\} \sim S^n$. Consider $p \colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $p(v) = \frac{v}{\|v\|}$, and $q \colon S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ defined by $q(v) = v$. Then $p \circ q = \mathrm{id}$, and $(q \circ p)(v) = \frac{v}{\|v\|}$. This is homotopic to the identity by

$$H(v,t) = \frac{v}{(1-t) + t\|v\|}$$

This is a special case of a *retract*, a continuous map onto a subspace.

3 Groups from loops

3.1 Homotopy relative to a set

Definition. Let $A \subseteq X$. We say $f_0, f_1 : X \to Y$ are *homotopic relative to* A, written $f_0 \sim f_1 \text{ rel } A$, if $f_0 \sim f_1$ via some homotopy $H : X \times I \to Y$ that fixes A, so $H(a, t) = f_0(a) = f_1(a)$ for all $a \in A$.

Lemma. Homotopy relative to A is an equivalence relation.

Lemma. If $f_0, f_1 : X \to Y$ and $f_0 \sim f_1 \text{ rel } A$, and $g_0, g_1 : Y \to Z$ and $g_0 \sim g_1 \text{ rel } f(A)$, then $g_0 \circ f_0 \sim g_1 \circ f_1 \text{ rel } A$.

If $\gamma_0, \gamma_1 : I \to X$ are two homotopic paths relative to their endpoints, so $\gamma_0 \sim \gamma_1 \text{ rel } \{0, 1\}$, we write $\gamma_0 \sim_e \gamma_1$.

Lemma. Let $f_0, f_1 : I \to I$, where $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$. Then $f_0 \sim_e f_1$.

Proof. I is convex, hence $H(x, t) = (1 - t)f_0(x) + tf_1(x)$ is a homotopy that preserves endpoints as required. □

Corollary. Suppose $f : I \to I, \gamma : I \to X$. Then if $f(0) = 0$ and $f(1) = 1$, $\gamma \circ f \sim_e \gamma$. Further, if $f(0) = 0$ and $f(1) = 0$, we have $\gamma \circ f \sim_e c_{I,\gamma(0)}$.

Proof. We have $f(0) = \text{id}_I(0)$ and $f(1) = \text{id}_I(1)$. Hence $f \sim_e \text{id}_I$. Therefore, $\gamma \circ f \sim_e \gamma \circ \text{id}_I = \gamma$.

For the second claim, $f(0) = c_{I,0}(0)$ and $f(1) = c_{I,0}(1)$, hence $f \sim_e c_{I,0}$ giving $\gamma \circ f \sim_e \gamma \circ c_{I,0} = c_{I,\gamma(0)}$. □

Definition. Let X be a space, and $p, q \in X$. Let

$$\Omega(X, p, q) = \{\gamma : I \to X \mid \gamma \text{ continuous}, \gamma(0) = p, \gamma(1) = q\}$$

be the set of paths from p to q. Let $\Omega(X, p) = \Omega(X, p, p)$ be the set of loops based at p.

Definition. Let $\gamma \in \Omega(X, p, q), \gamma' \in \Omega(X, q, r)$. Then their composition $\gamma\gamma' \in \Omega(X, p, r)$ is given by

$$(\gamma\gamma')(t) = \begin{cases} \gamma(2t) & t \in \left[0, \frac{1}{2}\right] \\ \gamma'(2t - 1) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$\gamma\gamma'$ is continuous by the gluing lemma.

Lemma. Let $\gamma_0, \gamma_1 \in \Omega(X, p, q)$ and $\gamma_0', \gamma_1' \in \Omega(X, q, r)$ such that $\gamma_0 \sim_e \gamma_1$ via $H : I \times I \to X$ and $\gamma_0' \sim_e \gamma_1'$ via $H' : I \times I \to X$. Then $\gamma_0\gamma_0' \sim_e \gamma_1\gamma_1'$.

Proof. The homotopy required is

$$\overline{H}(x,t) = \begin{cases} H(2x,t) & x \in \left[0, \frac{1}{2}\right] \\ H'(2x-1,t) & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

□

Definition. Let $\gamma \in \Omega(X, p, q)$. Then $\gamma^{-1} \in \Omega(X, q, p)$ is the *reverse* of γ, given by

$$\gamma^{-1}(t) = \gamma(1-t)$$

Proposition. (i) Let $\gamma \in \Omega(X, p, q)$. Then $c_{I,p}\gamma \sim_e \gamma \sim_e \gamma c_{I,q}$.
(ii) $\gamma\gamma^{-1} \sim_e c_{I,p}$ and $\gamma^{-1}\gamma \sim_e c_{I,q}$.
(iii) If $\gamma(1) = \gamma'(0)$ and $\gamma'(1) = \gamma''(0)$, we have

$$\gamma(\gamma'\gamma'') \sim_e (\gamma\gamma')\gamma''$$

Proof. (i) The composition $c_{I,p}\gamma$ has $c_{I,p}\gamma(t) = \gamma(f(t))$ where $f : I \to I$ defined by

$$f(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right] \\ 2t-1 & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Since $f(0) = 0$ and $f(1) = 1$, $\gamma \circ f \sim_e \gamma$. Similarly, $\gamma c_{I,q}(t) = \gamma(g(t))$ where

$$g(t) = \begin{cases} 2t & t \in \left[0, \frac{1}{2}\right] \\ 1 & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

(ii) $\gamma\gamma^{-1}(t) = \gamma(f(t))$ where

$$f(t) = \begin{cases} 2t & t \in \left[0, \frac{1}{2}\right] \\ 1-2t & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Further, $\gamma^{-1}\gamma(t) = \gamma(g(t))$ where

$$g(t) = \begin{cases} 1-2t & t \in \left[0, \frac{1}{2}\right] \\ 2t-1 & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

(iii) We can write $\gamma(\gamma'\gamma'')(t) = (\gamma\gamma')\gamma(f(t))$ where $f : I \to I$ is the continuous function defined by

$$f(t) = \begin{cases} \frac{t}{2} & t \in \left[0, \frac{1}{2}\right] \\ t - \frac{1}{4} & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2t-1 & t \in \left[\frac{3}{4}, 1\right] \end{cases}$$

noting that $f(0) = 0$ and $f(1) = 1$. Hence $\gamma(\gamma'\gamma'') \sim_e = (\gamma\gamma')\gamma''$.

□

Definition. Let X be a space and $x_0 \in X$. We define the *fundamental group* or *first homotopy group* of X based at x_0 by

$$\pi_1(X, x_0) = {}^{\Omega(X, x_0)}/_{\sim_e}$$

We say x_0 is the *basepoint*. If $\gamma \in \Omega(X, x_0)$, we write $[\gamma]$ for its image in $\pi_1(X, x_0)$, its equivalence class.

Theorem. We define multiplication in π_1 by $[\gamma] * [\gamma'] = [\gamma\gamma']$. The identity is $1 = [c_{I,x_0}]$. The inverse is given by $[\gamma]^{-1} = [\gamma^{-1}]$. These operations form a group.

Proof. Using the above lemma we explicitly check the group axioms. Identity:

$$1[\gamma] = [c_{I,x_0}\gamma] = [\gamma]; \quad [\gamma]1 = [\gamma c_{I,x_0}] = [\gamma]$$

Inverses:

$$[\gamma][\gamma]^{-1} = [\gamma\gamma^{-1}] = [c_{I,x_0}] = 1$$

Associativity:

$$([\gamma][\gamma'])[\gamma''] = [\gamma\gamma'][\gamma''] = [(\gamma\gamma')\gamma''] = [\gamma(\gamma'\gamma'')] = [\gamma][\gamma'\gamma''] = [\gamma]([\gamma'][\gamma''])$$

□

3.2 Induced maps

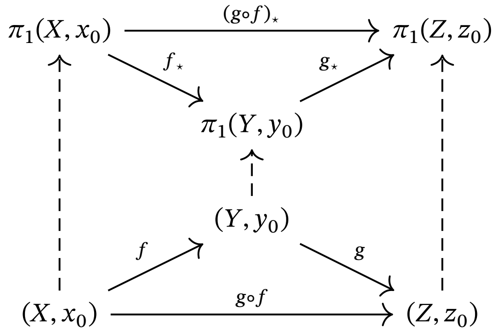
Definition. Let $f : X \to Y$ be a continuous map, and $f(x_0) = y_0$. Then we have a map $\Omega(X, x_0) \to \Omega(Y, y_0)$ defined by $\gamma \mapsto f \circ \gamma$. Note that if $\gamma_0 \sim_e \gamma_1$, we have $f \circ \gamma_0 \sim_e f \circ \gamma_1$. Thus, this map descends to the *induced homomorphism* $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $[\gamma] \mapsto [f \circ \gamma]$.

Definition. A *pointed space* (X, x_0) is a pair where X is a space and $x_0 \in X$. We write $f : (X, x_0) \to (Y, y_0)$ to denote a map $f : X \to Y$ where $f(x_0) = y_0$. In particular, for $f : (X, x_0) \to (Y, y_0)$ there is an induced map $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proposition. Let $f : (X, x_0) \to (Y, y_0)$. Then,

(i) The induced map $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group homomorphism.
(ii) $\left(\mathrm{id}_{(X,x_0)}\right)_\star = \mathrm{id}_{\pi_1(X,x_0)}$.
(iii) If $g : (Y, y_0) \to (Z, z_0)$, we have $(g \circ f)_\star = g_\star \circ f_\star$.
(iv) If $f_0, f_1 : (X, x_0) \to (Y, y_0)$ with $f_0 \sim f_1$ rel x_0, then $(f_0)_\star = (f_1)_\star$ (*homotopy invariance*).

Remark. The action of taking the fundamental group of a pointed space thus yields a functor π_1 : **Top.** → **Grp**. The following diagram, representing part (iii) of the proposition above, commutes.



Proof. (i) This follows from the fact that

$$f \circ (\gamma\gamma')(t) = \begin{cases} f \circ \gamma(2t) & t \in \left[0, \frac{1}{2}\right] \\ f \circ \gamma'(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases} = (f \circ \gamma)(f \circ \gamma')(t)$$

Hence,

$$f_\star([\gamma][\gamma']) = [f \circ (\gamma\gamma')] = [(f \circ \gamma)(f \circ \gamma')] = [f \circ \gamma][f \circ \gamma'] = f_\star([\gamma])f_\star([\gamma'])$$

(ii) $\mathrm{id}_\star([\gamma]) = [\mathrm{id}_X \circ \gamma] = [\gamma]$.

(iii) $(f \circ g)_\star([\gamma]) = [f \circ g \circ \gamma] = f_\star([g \circ \gamma]) = f_\star(g_\star([\gamma]))$.

(iv) $f_0 \sim f_1$ rel x_0 and $\gamma(0) = \gamma(1) = x_0$ implies $f_0 \circ \gamma \sim_e f_1 \circ \gamma$, so $(f_0)_\star([\gamma]) = (f_1)_\star([\gamma])$.

□

Example. Let $f : X \to Y$ be a homeomorphism, and let $y_0 = f(x_0)$. Then $f : (X, x_0) \to (Y, y_0)$ and $f^{-1} : (Y, y_0) \to (X, x_0)$ are inverses. Thus, $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_\star^{-1} : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ are inverses. Since $f_\star \circ f_\star^{-1} = (f \circ f^{-1})_\star = \mathrm{id}_{\pi_1(Y,y_0)}$ and $f_\star^{-1} \circ f_\star = \mathrm{id}_{\pi_1(X,x_0)}$, we have that $f_\star$ is a group isomorphism, and π_1 is a topological invariant.

3.3 Retractions

Definition. Let $A \subset X$, where $\iota : A \to X$ is the inclusion map. Then $p : X \to A$ is a *retraction* if $p \circ \iota = \mathrm{id}_A$. $p : X \to A$ is a *strong deformation retraction*, or *s.d.r.*, if $p \circ \iota = \mathrm{id}_A$ and $\iota \circ p \sim \mathrm{id}_X$ rel A.

Remark. In either case, if $a_0 \in A$, $\iota : (A, a_0) \to (X, a_0)$ and $p : (X, a_0) \to (A, a_0)$. If p is a retraction, $p_\star \circ \iota_\star = (p \circ \iota)_\star = (\mathrm{id}_A)_\star = \mathrm{id}_{\pi_1(A,a_0)}$, so $\iota_\star : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective, and $p_\star : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective. If p is a strong deformation retraction, $\iota_\star \circ p_\star = (\iota \circ p)_\star = (\mathrm{id}_X)_\star = \mathrm{id}_{\pi_1(X,a_0)}$, so $p_\star$ and $\iota_\star$ are isomorphisms.

Remark. If $p : X \to A$ is a strong deformation retraction, then $A \sim X$.

Example. $p : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by $v \mapsto \frac{v}{\|v\|}$ is a strong deformation retraction.

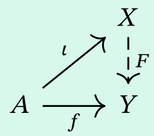
Example. $\mathbb{R}^2 \setminus \{0, 1\}$ has A, B as strong deformation retractions, where A is a figure-eight with one loop surrounding each hole, and B is a rectangle surrounding each hole with a vertical line connecting the top and bottom edges through $\left(\frac{1}{2}, 0\right)$. This can be a useful trick to show $A \sim B$.

3.4 Null-homotopy and extensions

Definition. We say $f: X \to Y$ is *null-homotopic* if $f \sim c_{X,p}$ for $p \in Y$.

Example. If X is contractible, then $\mathrm{id}_X \sim c_{X,q}$, so $f = f \circ \mathrm{id}_X \sim f \circ c_{X,q} = f(q)$. So any $f: X \to Y$ is null-homotopic. If $f_0 \sim f_1$, then f_0 is null-homotopic if and only if f_1 is null-homotopic.

Definition. Let $A \subset X$ and $f: A \to Y$. We say a continuous map $F: X \to Y$ is an *extension* of f if $F|_A = f$. If such a map exists, we say f *extends* to X.



Lemma. $f: S^1 \to Y$ extends to D^2 if and only if f is null-homotopic.

Proof. If F is an extension of f to D^2, we define $H(v,t) = F(tv)$. Then H is a homotopy from f to $c_{S^1,F(0)}$. So f is null-homotopic.

Conversely, if f is null-homotopic, let $H: S^1 \times I \to Y$ be a homotopy for $c_{S^1,p} \sim f$. Then we define

$$F(v) = \begin{cases} H\left(\frac{v}{\|v\|}, \|v\|\right) & v \neq 0 \\ p & v = 0 \end{cases}$$

One can check that this is indeed a continuous extension. □

Definition. Let $\gamma \in \Omega(X, x_0)$. We define $\overline{\gamma}: S^1 \to X$ by $\overline{\gamma}(e^{2\pi i t}) = \gamma(t)$. This is well-defined since $\gamma(0) = \gamma(1)$, and it is continuous because $I/\{0,1\} \simeq S^1$.

Lemma. (i) If $\gamma_0 \sim_e \gamma_1$ via $H(x,t)$, we have $\overline{\gamma}_0 \sim \overline{\gamma}_1$ via $\overline{H}: S^1 \times I \to Y$ given by $\overline{H}(e^{2\pi i x}, t) = H(x,t)$.
(ii) $\overline{\gamma\gamma'} \sim \overline{\gamma'\gamma}$.

Proof. (i) Note that $\overline{H}$ is well-defined since $H(0,t) = H(1,t) = x_0$.

(ii) We have $\overline{\gamma\gamma'}(v) = \overline{\gamma'\gamma}(-v)$, hence $\overline{\gamma\gamma'} = \overline{\gamma'\gamma} \circ a$ where $a: S^1 \to S^1$ is the antipodal map. Since $a \sim \mathrm{id}_{S^1}$, we have $\overline{\gamma\gamma'} \sim \overline{\gamma'\gamma}$.

□

Consider the radial projection homeomorphism $\Phi: D^2 \to I \times I$. Note that $\Phi(S^1) = \partial(I \times I) = I \times \{0,1\} \cup \{0,1\} \times I$. Since Φ is a homeomorphism, $h: \partial(I \times I) \to X$ extends to $I \times I$ if and only if

$h \circ \Phi$ extends to D^2, which is true if and only if $h \circ \Phi$ is null-homotopic. Define $\alpha_i(t) = h(t,i)$ and $\beta_i(t) = h(i,t)$ for $i = 0, 1$. Then, $h \circ \Phi \sim \overline{\alpha_0\beta_1\alpha_1^{-1}\beta_0^{-1}}$.

Proposition. Let $\gamma_0, \gamma_1 \in \Omega(X, p, q)$. Then the following are equivalent.
(i) $\gamma_0 \sim_e \gamma_1$;
(ii) $\overline{\gamma_0\gamma_1^{-1}}$ is null-homotopic;
(iii) $[\gamma_0\gamma_1^{-1}] = 1$ in $\pi_1(X, p)$.

Proof. Consider $h \colon \partial(I \times I) \to X$ given by $\gamma_0 c_{I,q}\gamma_1^{-1}c_{I,p}$. Note that h is continuous by the gluing lemma. $\gamma_0 \sim_e \gamma_1$ if and only if h extends to $I \times I$, which is true if and only if $h \circ \Phi$ extends to D^2, if and only if $\overline{\gamma_0 c_{I,q}\gamma_1^{-1}c_{I,p}}$ is null-homotopic. But this is homotopic to $\overline{\gamma_0\gamma^{-1}}$, so this proves that (i) and (ii) are equivalent.

Now, consider $h' \colon \partial(I \times I) \to X$ given by $\gamma_0\gamma_1^{-1}$ on one side, and on all other sides, $c_{I,p}$. Then $[\gamma_0\gamma_1^{-1}] = 1$ if and only if $\gamma_0\gamma_1^{-1} \sim_e c_{I,p}$, if and only if h' extends to $I \times I$, if and only if $h \circ \Phi$ extends to D^2, if and only if $\overline{\gamma_0\gamma_1^{-1}c_{I,p}c_{I,p}^{-1}c_{I,p}^{-1}} \sim \overline{\gamma_0\gamma_1^{-1}}$ is null-homotopic. ☐

Corollary. The following are equivalent.
(i) $\gamma_0 \sim_e \gamma_1$ for all $\gamma_0, \gamma_1 \in \Omega(X, p, q)$ and all $p, q \in X$.
(ii) any $f \colon S^1 \to X$ is null-homotopic;
(iii) $\pi_1(X, x_0)$ is the trivial group for all $x_0 \in X$.

Definition. X is *simply connected* if X is path-connected and $\pi_1(X, x_0) = 1$ for all $x_0 \in X$.

3.5 Change of basepoint

Lemma. Let X_0 be the path-connected component of X containing a point $x_0 \in X$. If Z is path-connected, $f \colon Z \to X$ is continuous, and $x_0 \in \operatorname{Im} f$, we have $\operatorname{Im} f \subseteq X_0$.

Proof. Suppose $f(z_0) = x_0$. Given $z \in Z$, choose $\gamma \in \Omega(Z, z_0, z)$ by path-connectedness. Then $f \circ \gamma \in \Omega(X, x_0, f(z))$, so $f(Z) \subseteq X_0$. ☐

Let $\iota \colon (X_0, x_0) \to (X, x_0)$ be the inclusion map. Then if $f \colon (Z, z_0) \to (X, x_0)$ and Z is path-connected, f factors through ι as $f = \iota \circ \hat{f}$ where $\hat{f} \colon (Z, z_0) \to (X_0, x_0)$.

Lemma. The map $\iota_\star \colon \pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$, so $\gamma \colon (I, 0) \to (X, x_0)$ giving $\gamma = \iota \circ \hat{\gamma}$ where $\hat{\gamma} \in \Omega(X_0, x_0)$; $[\gamma] = \iota_\star([\hat{\gamma}])$, so $\iota_\star$ is surjective. Now suppose $\gamma_0 = \iota \circ \hat{\gamma}_0, \gamma_1 = \iota \circ \hat{\gamma}_1$. If $\iota_\star([\hat{\gamma}_0]) = \iota_\star([\hat{\gamma}_1])$, so $\gamma_0 \sim_e \gamma_1$ via $H \colon I \times I \to X$, we have $H(0, 0) = x_0$, so $H = \iota \circ \hat{H}$ since $I \times I$ is path-connected. Then we can check $\hat{H}$ is a homotopy for $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$. Hence $[\hat{\gamma}_0] = [\hat{\gamma}_1]$, so $\iota_\star$ is injective. ☐

Let $u \in \Omega(X, x_0, x_1)$. Then we can define $u_\sharp : \Omega(X, x_0) \to \Omega(X, x_1)$ by $\gamma \mapsto u^{-1}\gamma u$. Hence if $\gamma_0 \sim_e \gamma_1$, we have $u^{-1}\gamma_0 u \sim_e u^{-1}\gamma_1 u$, so $u_\sharp$ descends to a map $u_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by $[\gamma] \mapsto [u^{-1}\gamma u]$.

Proposition. $u_\sharp$ is a group isomorphism with inverse $(u^{-1})_\sharp$.

$$\pi_1(X, x_0) \underset{(u^{-1})_\sharp}{\overset{u_\sharp}{\rightleftarrows}} \pi_1(X, x_1)$$

Proof. First, it is a homomorphism.

$$\begin{aligned} u_\sharp([\gamma][\gamma']) &= [u^{-1}\gamma\gamma' u] = [u^{-1}\gamma c_{I,x_0}\gamma' u] \\ &= [u^{-1}\gamma u u^{-1}\gamma' u] = [u^{-1}\gamma u][u^{-1}\gamma' u] = u_\sharp([\gamma])u_\sharp([\gamma']) \end{aligned}$$

Consider the function $u_\sharp^{-1}$. We have

$$u_\sharp^{-1}(u_\sharp([\gamma])) = [uu^{-1}\gamma u u^{-1}] = [c_{I,x_0}\gamma c_{I,x_0}] = [\gamma]$$

and

$$u_\sharp(u_\sharp^{-1}([\gamma])) = [u^{-1}u\gamma u^{-1}u] = [c_{I,x_1}\gamma c_{I,x_1}] = [\gamma]$$

So $u_\sharp, u_\sharp^{-1}$ are inverses, and therefore isomorphisms. $\square$

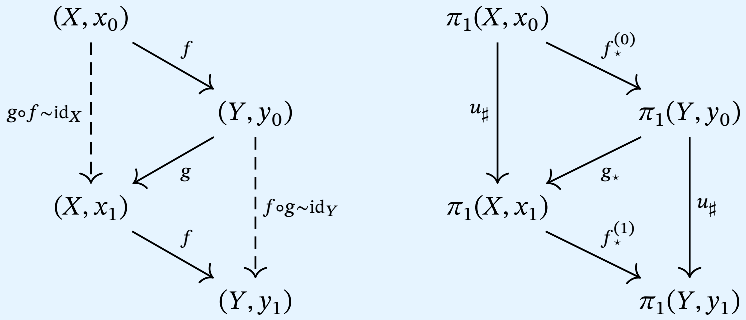
Corollary. A space X is simply connected if it is path-connected and $\pi_1(X, x_0) = 1$ for any $x_0 \in X$, since then it follows that $\pi_1(X, x) = 1$ for all $x \in X$.

Theorem. Let $x_0 \in X$, and $f_0, f_1 : X \to Y$ such that $f_0 \simeq f_1$ by $H : X \times I \to Y$. Let $u(t) = H(x_0, t)$ and $y_0 = f_0(x_0), y_1 = f_1(x_0)$. Then $u \in \Omega(Y, y_0, y_1)$. We have $f_i : (X, x_0) \to (Y, y_i)$ which induce $f_{i\star} : \pi_1(X, x_0) \to \pi_1(Y, y_i)$. Then $f_{1\star} = u_\sharp \circ f_{0\star}$.

$$\begin{array}{ccc} & & (Y, y_0) \\ & \overset{f_0}{\nearrow} & \\ (X, x_0) & & \\ & \underset{f_1}{\searrow} & \\ & & (Y, y_1) \end{array} \qquad \begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \overset{f_{0\star}}{\nearrow} & \big\downarrow u_\sharp \\ \pi_1(X, x_0) & & \\ & \underset{f_{1\star}}{\searrow} & \\ & & \pi_1(Y, y_1) \end{array}$$

Proof. We must show that $f_{1\star}([\gamma]) = u_\sharp(f_{0\star}([\gamma]))$. Let $\gamma_i = f_i \circ \gamma$. We therefore need to show $\gamma_1 \sim_e u^{-1}\gamma_0 u$ for all $\gamma \in \Omega(X, x_0)$. Suppose we can show that $H : \partial(I \times I) \to Y$ given by $\gamma_0, u, \gamma_1^{-1}, u^{-1}$ on each side of the square extends to $I \times I$. Equivalently, $\overline{\gamma_0 u \gamma_1^{-1} u^{-1}} = \overline{u^{-1}\gamma_0 u \gamma_1^{-1}}$ is null-homotopic. This is equivalent to the statement $u^{-1}\gamma_0 u \sim_e \gamma_1$. We know h extends to $\hat{H} : I \times I \to Y$, because $\hat{H}(x, t) = H(\gamma(x), t)$. $\square$

Corollary. Let $X \sim Y$ via $f : X \to Y$ and $g\colon Y \to X$, so $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. Let $x_0 \in X$ and $f(x_0) = y_0$. Let $g(y_0) = x_1$ and $f(x_1) = y_1$. Then we have induced maps $f_\star^{(0)} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $g_\star : \pi_1(Y, y_0) \to \pi_1(X, x_1)$, $f_\star^{(1)} : \pi_1(X, x_1) \to \pi_1(Y, y_1)$. Then $g_\star$ is an isomorphism.



The left-hand commutative diagram, in the category of pointed topological spaces, commutes up to homotopy. The right-hand induced diagram commutes.

Proof. We have $\mathrm{id}_X \sim g \circ f$ via $H\colon X \times I \to X$. Then $g_\star \circ f_\star^{(0)} = (g \circ f)_\star = u_\sharp \circ (\mathrm{id}_X)_\star$ where $u(t) = H(x_0, t)$ is a path from x_0 to x_1. Since $u_\sharp$ is an isomorphism, $g_\star$ is surjective. Similarly, $f^{(1)} \circ g_\star = (f \circ g)_\star$ is an isomorphism, so $g_\star$ is injective. ☐

Corollary. Let X be contractible. Then $\pi_1(X, x_0) = 1$ is the trivial group.

Proof. The space $\Omega(\{\bullet\}, \bullet)$ has one element, so $\pi_1(\{\bullet\}, \bullet) = 1$. Since $X \sim \{\bullet\}$, the result follows. ☐

4 Covering spaces

4.1 Definitions

Definition. Let $p : \hat{X} \to X$ be a continuous function. We say $U \subset X$ is *evenly covered* by p if $p^{-1}(U) \simeq \coprod_{\alpha \in A} U_\alpha$ and $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for all α.

The topology on the coproduct $\coprod_{\alpha \in A} U_\alpha$ is such that V is open if and only if each projection $V \cap U_\alpha$ is open. The topology on $p^{-1}(U)$ is the subspace topology. In particular, the inclusions $\iota_\alpha\colon U_\alpha \to \coprod_{\alpha \in A} U_\alpha \to \hat{X}$ are continuous, as is the composition $\iota_\alpha\big(p|_{U_\alpha}\big)^{-1} : U \to X$ since $p|_{U_\alpha}$ is a homeomorphism.

Definition. $p\colon \hat{X} \to X$ is a *covering map* if every $x \in X$ has an open neighbourhood U_x which is evenly covered by p. If so, we say $\hat{X}$ is a *covering space* of X.

Example. If A is a space with the discrete topology, then $p\colon A \times X \to X$ is a covering map, because $p^{-1}(X) = \coprod_{\alpha \in A} \{\alpha\} \times X$.

Example. $p\colon \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$ is a covering map. Indeed, if $V \subseteq \mathbb{R}$ is an open interval of at most unit length, let $U = p(V)$ and then $p^{-1}(U) = \coprod_{n \in \mathbb{Z}} V_n$ for $V_n = \{n + v \mid v \in V\}$.

Example. Consider $p_n\colon S^1 \to S^1$ defined by $z \mapsto z^n$. If $V \subseteq S^1$ is an open interval of length $< \frac{2\pi}{n}$, let $U = p_n(V)$. Then $p_n^{-1}(V) = \coprod_{i \in \mathbb{Z}/n\mathbb{Z}} \omega^i V$ for $\omega = e^{\frac{2\pi i}{n}}$. Hence U is evenly covered.

Definition. We define the n-dimensional real projective space as $\mathbb{RP}^n = S^n/\sim$ where $\sim$ is the equivalence relation generated by $x \sim -x$ for all $x \in S^n$.

Example. The quotient map $p\colon S^n \to \mathbb{RP}^n$ is a covering map. Indeed, for $x \in S^n$, let V_x be the open hemisphere centred at x. Then letting $U_x = p(V_x)$, we have $p^{-1}(U(x)) = U_x \amalg -U_x$, giving that U_x is evenly covered.

4.2 Lifting paths and homotopies

Definition. Let $p\colon \hat{X} \to X$ be a covering map, and $f\colon Z \to X$ be continuous. A continuous function $\hat{f}\colon Z \to \hat{X}$ is a *lift* if $p \circ \hat{f} = f$. Hence, the following commutative diagram holds.

$$\begin{array}{ccc} & & \hat{X} \\ & \overset{\hat{f}}{\nearrow} & \downarrow p \\ Z & \xrightarrow[f]{} & X \end{array}$$

Theorem (Path lifting)**.** Let $p\colon (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map, and $\gamma\colon [a, b] \to X$ be a path. Let $\gamma(a) = x_0$ and $p(\hat{x}_0) = \hat{x}_0$. Then there exists a unique lift $\hat{\gamma}\colon [a, b] \to \hat{X}$ with $\hat{\gamma}(a) = \hat{x}_0$.

The proof will be given after some lemmas. We say $f\colon Z \to X$ has the *(unique) lifting property at $z \in Z$* if for any $\hat{x} \in \hat{X}$ such that $p(\hat{x}) = f(z)$, there exists a (unique) lift $\hat{f}\colon Z \to \hat{X}$ such that $\hat{f}(z) = \hat{x}$.

Lemma (Lebesgue covering lemma)**.** Let X be a compact metric space, and $\{U_\alpha \mid \alpha \in A\}$ is an open cover of X. Then there exists $\delta > 0$ such that for every $x \in X$, the open ball $B_\delta(x)$ is contained in U_α for some $\alpha \in A$.

Proof. We have an open cover $\{U_\alpha \mid \alpha \in A\}$ of X, so given $x \in X$, we can find $\alpha_x \in A$ such that $x \in U_{\alpha_x}$ and U_{α_x} is open. Hence there exists $\delta_x > 0$ such that $B_{2\delta_x}(x) \subset U_{\alpha_x}$. Then $\{B_{\delta_x}(x) \mid x \in X\}$ is an open cover of X. By compactness there is a finite subcover $\{B_{\delta_{x_i}}(x_i) \mid i \in \{1, \dots, k\}\}$. Let $\delta = \min_{i \in \{1,\dots,k\}} \delta_{x_i} > 0$. Then for $y \in X$, we have $y \in B_{\delta_{x_i}}(x_i)$ for some i, and $B_\delta(y) \subset B_{\delta_{x_i}+\delta}(x_i) \subset B_{2\delta_{x_i}}(x_i) \subset U_{\alpha_x}$. $\square$

Lemma. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map, and $\gamma : [a, b] \to X$ be a path such that $\gamma(a) = x_0$. Let $\operatorname{Im}\gamma \subset U$ where $U \subset X$ is evenly covered. Then γ has the unique lifting property.

Note that this is simply the above path lifting theorem with an additional hypothesis.

Proof. Since U is evenly covered, $p^{-1}(U) = \coprod_{\alpha \in A} U_\alpha$, and $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism onto its image. So $\hat{x}_0 \in U_{\alpha_0}$ for some $\alpha_0 \in A$. Then the map $(p_\alpha)^{-1} = \iota_\alpha \circ \left(p|_{U_\alpha} \right)^{-1} : U \to \hat{X}$ is continuous. Then $\left(p|_{U_{\alpha_0}} \right)^{-1}(x_0) = \hat{x}_0$, so $\hat{\gamma} = \left(p_{\alpha_0}\right)^{-1} \circ \gamma$ is a lift of γ with $\hat{\gamma}(a) = \hat{x}_0$.

Now we will prove uniqueness of the lift. Observe that $p^{-1}(U) = U_{\alpha_0} \amalg \coprod_{\alpha \neq \alpha_0} U_\alpha$ disconnects $p^{-1}(U)$. Note that $[a, b]$ is connected. We have that if $\hat{\gamma} : [a, b] \to \hat{X}$ with $\hat{\gamma}(a) = \hat{x}_0$ and $p \circ \hat{\gamma} = \gamma$, then $\operatorname{Im}\hat{\gamma} \subset p^{-1}(U)$ implies $\operatorname{Im}\hat{\gamma} \subset U_{\alpha_0}$. But $p|_{U_{\alpha_0}}$ is a homeomorphism, so we must have $\hat{\gamma} = \left(p_{\alpha_0}\right)^{-1} \circ \gamma$. □

Lemma. Let $\gamma : [a, b] \to X$ and $a' \in [a, b]$. If $\gamma|_{[a,a']}$ has the unique lifting property at a and $\gamma|_{[a',b]}$ has the unique lifting property at a', then γ has the unique lifting property at a.

Proof. If $p(\hat{x}) = \gamma(a)$, since $\gamma|_{[a,a']}$ has the unique lifting property at a, there exists a unique lift $\hat{\gamma}_1 : [a, a'] \to \hat{X}$ such that $\hat{\gamma}_1(a) = \hat{x}$. Then $\gamma|_{[a',b]}$ has the unique lifting property at a', so there exists a unique lift $\hat{\gamma}_2 : [a', b] \to \hat{X}$ with $\hat{\gamma}_2(a') = \hat{\gamma}_1(a')$. Then the composition $\hat{\gamma} = \hat{\gamma}_1\hat{\gamma}_2$ is a lift of γ, with $\hat{\gamma}(a) = \hat{x}$.

For uniqueness, suppose $\hat{\gamma}$ is a lift of γ with $\hat{\gamma}(a) = \hat{x}$. Then $\hat{\gamma}|_{[a,a']}$ is a lift of $\gamma|_{[a,a']}$, so by the unique lifting property, $\hat{\gamma}|_{[a,a']}$ is uniquely determined such that $\hat{\gamma}(a) = \hat{x}$. Then by the unique lifting property again, $\hat{\gamma}|_{[a',b]}$ is also uniquely determined such that $\hat{\gamma}|_{[a',b]}(a') = \hat{\gamma}|_{[a,a']}(a')$. □

We can now prove the path lifting theorem: any $\gamma : I \to X$ has the unique lifting property.

Proof. Let $p : \hat{X} \to X$ be a covering map. Hence, for all $x \in X$, there exists an open neighbourhood U_x which is evenly covered. $\{U_x \mid x \in X\}$ is therefore an open cover of X, and so $\left\{\gamma^{-1}(U_x) \mid x \in X\right\}$ is an open cover of I. Since I is compact, by the Lebesgue covering lemma, there exists $\delta > 0$ such that for all t, $B_\delta(t) \subseteq \gamma^{-1}(U_{x(t)})$ for some $x(t)$. In other words, $\gamma(B_\delta(t)) \subseteq U_{x(t)}$.

Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$, and $a_i = \frac{i}{n} \in I$. Then $[a_i, a_{i+1}] \subset B_\delta(a_i)$ for all i. Hence $\gamma[a_i, a_{i+1}] \subseteq U_{x(a_i)}$. Then $[a_i, a_{i+1}]$ is connected, hence $\gamma[a_i, a_{i+1}]$ is connected. Since $U_{x(a_i)}$ is evenly covered, $\gamma|_{[a_i,a_{i+1}]}$ has the unique lifting property. Then by induction on i, we can see that $\gamma|_{[0,a_i]}$ has the unique lifting property, and hence so does γ in its entirety. □

Theorem (Homotopy lifting). Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map, and $H : I \times I \to X$ be a homotopy. Then H has the lifting property at $(0, 0)$.

It also has the unique lifting property, but this will be more easily proven later.

Proof. I is compact and connected, so by Tychonoff's theorem, $I \times I$ is compact and connected. Suppose $\{U_x \mid x \in X\}$ is an open cover of X consisting of evenly covered neighbourhoods of points as before. Then, since $I \times I$ is compact, by the Lebesgue covering lemma there exists $\delta > 0$ such that for all $v \in I \times I$, $B_\delta(v) \subseteq H^{-1}(U_{x(v)})$. In particular, $H(B_\delta(v)) \subseteq U_{x(v)}$.

Let $n \in \mathbb{N}$ such that $\frac{\sqrt{2}}{n} < \delta$, dividing $I \times I$ into squares of size $\frac{1}{n}$, ordered from left-to-right and then bottom-to-top. Label each square with an index $i \in \{1, \dots, n^2\}$. Let each square A_i have lower left-hand corner v_i, for $i \in \{1, \dots, n^2\}$. Note that $H(A_i) \subseteq H(B_\delta(v_i)) \subseteq U_{x(v_i)} = U_i$ is evenly covered.

Let $B_k = \bigcup_{i=1}^k A_i$. Then $A_i \simeq I \times I$ is connected, so $H|_{A_i}$ has the lifting property at v_i.

We show by induction that $H|_{B_k}$ has the lifting property at $(0,0)$. For $k = 1$, $B_1 = A_1$ and $(0,0) = v_1$, so the result follows.

For other k, suppose that $H|_{B_k}$ has the lifting property at $(0,0)$, so $\hat{H}_k : B_k \to \hat{X}$ with $\hat{H}_k(0,0) = \hat{x}$. Then $H|_{A_{k+1}}$ has the lifting property at v_i, so choose a lift $\hat{h}_k : A_{k+1} \to \hat{X}$ such that $\hat{h}_k(v_{k+1}) = \hat{H}_k(v_{k+1})$. Note that $p(\hat{H}_k(v_{k+1})) = H(v_{k+1})$, so this exists by the lifting property. Observe that $A_{k+1} \cap B_k = I_k \cup I'_k$ is the union of (at most) two intervals with intersection at their endpoints, so is homeomorphic to I. Hence by uniqueness of path lifting, $\hat{H}_k|_{I_k} = \hat{h}_k|_{I_k}$ since both are lifts of $H|_{I_k}$ with $v_{k+1} \mapsto \hat{H}_k(v_{k+1})$. Similarly, $\hat{H}_k|_{I'_k} = \hat{h}_k|_{I'_k}$. In other words, $\hat{H}_k|_{A_{k+1} \cap B_k} = \hat{h}_k|_{A_{k+1} \cap B_k}$. By the gluing lemma, we can construct the well-defined and continuous map $\hat{H}_{k+1} : B_{k+1} \to X$ given by $\hat{H}_k$ and $\hat{h}_k$ on their domains. Then $\hat{H}_{k+1}$ is a lift of $H|_{B_{k+1}}$. □

Proposition. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. Let $\gamma_0, \gamma_1 \in \Omega(X, x_0, x_1)$, and $\gamma_0 \sim_e \gamma_1$. Let $\hat{\gamma}_i$ be the lift of γ_i to $\hat{X}$ with $\hat{\gamma}_i(0) = \hat{x}_0$, which exists by the path lifting property. Then $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$.

Proof. Let $H : I \times I \to X$ be a homotopy between γ_0 and γ_1. By the homotopy lifting property, there exists a lifted homotopy $\hat{H} : I \times I \to \hat{X}$ such that $\hat{H}(0,0) = \hat{x}_0$. Let $\alpha_i(t) = \hat{H}(t,i)$ for $i = 0,1$, and $\beta_i(t) = \hat{H}(i,t)$ for $i = 0,1$. Applying the uniqueness of path lifting to the α_i and the β_i,

(i) α_0 is a lift of γ_0 with $\alpha_0(0) = \hat{x}_0$, so $\alpha_0 = \hat{\gamma}_0$;

(ii) β_0 is a lift of c_{I,x_0} with $\beta_0(0) = \hat{x}_0$, so $\beta_0 = \hat{c}_{I,x_0} = c_{I,\hat{x}_0}$ by uniqueness, and in particular, $\alpha_1(0) = \beta_0(1) = \hat{x}_0$;

(iii) α_1 is a lift of γ_1 with $\alpha_1(0) = \hat{x}_0$, so $\alpha_1 = \hat{\gamma}_1$;

(iv) let $\hat{x}_1 = \hat{\gamma}_0(1)$, and then β_1 is a lift of c_{I,x_1}, so $\beta_1(0) = \hat{x}_1$, so $\beta_1 = c_{I,\hat{x}_1}$.

Hence $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$ via $\hat{H}$. □

Corollary. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. Let $\gamma_0, \gamma_1 \in \Omega(X, x_0, x_1)$, and $\gamma_0 \sim_e \gamma_1$. Then $\hat{\gamma}_0(1) = \hat{\gamma}_1(1)$.

4.3 Simply connected lifting

Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. If $\gamma : I \to X$ has $\gamma(0) = x_0$, let $\hat{\gamma} : I \to \hat{X}$ be its unique lift such that $\hat{\gamma}(0) = \hat{x}_0$.

Consider $\widehat{\gamma\gamma'} = \hat{\gamma}\tilde{\gamma}'$, where $\tilde{\gamma}'$ is a lift of γ' such that $\tilde{\gamma}'(0) = \hat{\gamma}(1)$. Note that we needed to change the start point of $\tilde{\gamma}'$ in the covering space.

Definition. A space X is *locally path-connected* if for every open set $U \subseteq X$ and $x \in U$, there exists an open $V \subseteq U$ with $x \in V$ and V path-connected.

Example. Consider

$$X = \{(x, 0) \in \mathbb{R}^2\} \cup \left\{\left(\frac{1}{n}, y\right) \in \mathbb{R}^2, n \in \mathbb{Z}\right\} \cup \{(0, y) \in \mathbb{R}^2\}$$

Then, an open set containing a point $(0, y)$ but not $(0, 0)$ admits no smaller path-connected open neighbourhood.

Proposition (simply connected lifting property)**.** Let Z be a simply connected (and hence path-connected) space that is also locally path-connected. If $f : (Z, z_0) \to (X, x_0)$, then f has a unique lift $\hat{f} : (Z, z_0) \to (\hat{X}, \hat{x}_0)$.

Remark. This proposition then implies the path lifting and homotopy lifting properties.

Proof. Suppose $\hat{f} : (Z, z_0) \to (\hat{X}, \hat{x}_0)$ is a lift of f. Given $z \in Z$, consider a path $\gamma \in \Omega(Z, z_0, z)$, which exists since Z is path-connected. Then $\hat{f} \circ \gamma$ is a lift of $f \circ \gamma$, since $p(\hat{f} \circ \gamma) = (p \circ \hat{f}) \circ \gamma = f \circ \gamma$. Then, $(\hat{f} \circ \gamma)(0) = \hat{f}(z_0) = \hat{x}_0$, so $\hat{f} \circ \gamma = \widehat{f \circ \gamma}$ is the unique lift of $f \circ \gamma$ given by the unique path lifting property. Then $\hat{f}(z) = \hat{f}(\gamma(1)) = (\hat{f} \circ \gamma)(1) = \widehat{f \circ \gamma}(1)$ is uniquely determined by the unique path lifting property. So any such lift is unique.

If $\gamma_0, \gamma_1 \in \Omega(Z, z_0, z)$, $\gamma_0 \sim_e \gamma_1$ by simply-connectedness. In particular, $f \circ \gamma_0 \sim_e f \circ \gamma_1$, and by the homotopy lifting property, $\widehat{f \circ \gamma_0}(1) = \widehat{f \circ \gamma_1}(1)$. So the choice of path γ used above is not relevant. Now, let us define $\hat{f} : (Z, z_0) \to (\hat{X}, \hat{x}_0)$ by $\hat{f}(z) = \widehat{f \circ \gamma}(1)$ where $\gamma \in \Omega(Z, z_0, z)$ is any path from z_0 to z. Then $p(\hat{f}(z)) = p \circ \widehat{f \circ \gamma}(1) = f \circ \gamma(1) = f(z)$ since $\widehat{f \circ \gamma}$ is a lift of $f \circ \gamma$. Hence $\hat{f}$ as defined is a lift. If $z = z_0$, we can take $\gamma = c_{I, z_0}$, so $f \circ \gamma = c_{I, x_0}$. In particular, $\widehat{f \circ \gamma} = c_{I, \hat{x}_0}$, so $\hat{f}(z) = \widehat{f \circ \gamma}(1) = \hat{x}_0$ as required.

Now, it suffices to check that $\hat{f}$ is a continuous function. Let $U \subseteq \hat{X}$ be an open neighbourhood of $\hat{f}(z)$. We need to find an open neighbourhood $V \subseteq Z$ of z such that $\hat{f}(V) \subseteq U$.

First, we find a subset $U' \subset U$ with $\hat{f}(z) \in U'$ such that $p(U')$ is open and evenly covered. Since p is a covering map, there exists an open $W \subseteq X$ with $f(z) \in W$ and which is evenly covered. Hence $p^{-1}(W) = \coprod_{\alpha \in A} W_\alpha$, and $p(\hat{f}(z)) = f(z)$, so $\hat{f}(z) \in W_{\alpha_0}$ for some $\alpha_0 \in A$. Then, $W_{\alpha_0} \subseteq \hat{X}$ is an open set. Let $U' = U \cap W_{\alpha_0}$. Then $\hat{f}(z) \in U'$, and $p|_{W_{\alpha_0}} : W_{\alpha_0} \to W$ is a homeomorphism, so $p(U') = p_{\alpha_0}(U')$ is open and evenly covered.

Next, $f : Z \to X$ is continuous, so we need to find an open set $V' \subseteq Z$ with $z \in V'$ and $f(V') \subseteq p(U')$. Since Z is locally path-connected, there exists $V \subseteq V'$ which is an open path-connected set with $z \in V$.

Now we need to show V satisfies the continuity requirement, that $\hat{f}(V) \subseteq U$. Given $z' \in V$, let $\gamma' \in \Omega(V, z, z')$, which exists because V is path-connected. Then $\operatorname{Im} f \circ \gamma' \subseteq f(V) \subseteq p(U')$. Note that $\operatorname{Im} f \circ \gamma'$ is evenly covered. Hence $\tilde{\gamma}' = p_{\alpha_0}^{-1} \circ f \circ \gamma'$ is a lift of $f \circ \gamma'$ with $\tilde{\gamma}'(0) = p_{\alpha_0}^{-1}(f(z)) = \hat{f}(z)$. Then $\gamma\gamma' \in \Omega(Z, z_0, z')$, and $\widehat{f \circ (\gamma\gamma')} = \widehat{f \circ \gamma}\tilde{\gamma}'$ by the discussion at the beginning of the subsection. Hence $\hat{f}(z') = \widehat{f \circ (\gamma\gamma')}(1) = \tilde{\gamma}'(1) = p_{\alpha_0}^{-1} \circ f \circ \gamma'(1) \in U'$. So $\hat{f}(V) \subseteq U$ as required. □

4.4 Universal covers

Let $p\colon (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. If $\gamma \in \Omega(X, x_0)$, let $\hat{\gamma}\colon I \to \hat{X}$ be its unique lift such that $\hat{\gamma}(0) = \hat{x}_0$, which exists by the path lifting property. Then there is a map $\varepsilon_p\colon \Omega(X, x_0) \to p^{-1}(x_0)$ by $\gamma \mapsto \hat{\gamma}(1)$, since $p(\hat{\gamma}(1)) = \gamma(1) = x_0$. By the corollary above, if $[\gamma_0] = [\gamma_1]$ in π_1, we have $\varepsilon_p(\gamma_0) = \varepsilon_p(\gamma_1)$. In particular, ε_p descends to a well-defined map from $\pi_1(X, x_0)$ to $p^{-1}(x_0)$.

Definition. A covering map $p\colon \hat{X} \to X$ is a *universal cover* if $\hat{X}$ is simply connected.

Example. $p\colon \mathbb{R} \to S^1$ defined by $x \mapsto e^{2\pi i x}$ is a universal cover of S^1, since $\mathbb{R}$ is contractible. $p_2\colon \mathbb{R}^2 \to S^1 \times S^1 = T^2$ defined by $p_2(x, y) = (p(x), p(y))$ is a universal cover.

Proposition. If $p\colon (\hat{X}, \hat{x}_0) \to (X, x_0)$ is a universal cover, then $\varepsilon_p\colon \pi_1(X, x_0) \to p^{-1}(x_0)$ is a bijection of sets.

Proof. Suppose $\varepsilon_p[\gamma_0] = \hat{x}_1 = \varepsilon_p[\gamma_1]$. Then $\hat{\gamma}_0$ and $\hat{\gamma}_1$ are paths in $\Omega(\hat{X}, \hat{x}_0, \hat{x}_1)$. Since $\hat{X}$ is simply connected, $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$. In particular, $\gamma_0 = p \circ \hat{\gamma}_0 \sim_e p \circ \hat{\gamma}_1 = \gamma_1$. Hence $[\gamma_0] = [\gamma_1]$, so ε_p is injective.

Given $\hat{x} \in p^{-1}(x_0)$, $\hat{X}$ is path-connected as it is simply connected, so there exists a path $\eta \in \Omega(\hat{X}, \hat{x}_0, \hat{x})$. Since $p(\hat{x}) = x_0$, we find $\gamma = p \circ \eta \in \Omega(X, x_0)$. Then $\eta = \hat{\gamma}$ is the unique lift of γ. In particular, $\varepsilon_p(\gamma) = \eta(1) = \hat{x}$, so ε_p is surjective. □

Example. Let $p\colon (\mathbb{R}, 0) \to (S^1, 1)$ be defined by $x \mapsto e^{2\pi i x}$. We have $p^{-1}(1) = \mathbb{Z}$. Then, $\varepsilon\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is a bijection.

Theorem. $\varepsilon_p\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is an isomorphism of groups.

Proof. It is a bijection, so it suffices to check that it is a homomorphism. Given $n \in \mathbb{Z}$, we can define $\varphi_n\colon \mathbb{R} \to \mathbb{R}$ by $\varphi_n(x) = x + n$. Then, $p \circ \varphi_n = p$. If $\gamma \in \Omega(S^1, 1)$, we can find a lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = 0$. Then $p \circ \varphi_n \circ \hat{\gamma} = p \circ \hat{\gamma} = \gamma$, so $\varphi_n \circ \hat{\gamma}$ is a lift of γ with $\varphi_n \circ \hat{\gamma}(0) = n$.

Suppose $\varepsilon_p[\gamma] = n$, and $\varepsilon_p[\gamma'] = n'$. Then $\hat{\gamma}(1) = n$, $\hat{\gamma}'(1) = n'$, so $\varphi_n \circ \hat{\gamma}'$ is a lift of γ' that starts at n. Hence, $\widehat{\gamma\gamma'} = \hat{\gamma}(\varphi_n \circ \hat{\gamma}')$ is a lift of the composition of paths. Thus, $\varepsilon[\gamma\gamma'] = \widehat{\gamma\gamma'}(1) = \varphi_n(\hat{\gamma}'(1)) = n + n'$. So ε_p is a homomorphism. □

Corollary. S^1 is not contractible.

Example. Let $f : S^1 \to S^1$ be the identity map. Let $p : (\mathbb{R}, 0) \to (S^1, 1)$ be a covering map. Then there is no lift of f to $\mathbb{R}$. Otherwise, the identity map on $\mathbb{Z}$ would factor through the trivial group. This shows that the simply connected lifting property does not extend to all path-connected spaces.

4.5 Degree of maps on the circle

Lemma. Let $z \in S^1$, and $u, v \in \Omega(S^1, z, 1)$. Then, the isomorphisms $u_\sharp, v_\sharp : \pi_1(S^1, z) \to \pi_1(S^1, 1)$ are equal.

Proof. Consider $v_\sharp^{-1} \circ u_\sharp = (v^{-1})_\sharp \circ u_\sharp$. Note, $(v_\sharp^{-1} \circ u_\sharp)[\gamma] = [vu^{-1}\gamma uv^{-1}]$. Since $vu^{-1} \in \Omega(S^1, 1)$, we can write $[vu^{-1}\gamma uv^{-1}] = [\eta][\gamma][\eta^{-1}]$ where $\eta = vu^{-1}$. But this is exactly $[\gamma]$, since $\pi_1(S^1, 1) \simeq \mathbb{Z}$ is abelian. Hence $v_\sharp^{-1} \circ u_\sharp = \mathrm{id}$, and by symmetry, $u_\sharp^{-1} \circ v_\sharp = \mathrm{id}$. □

Definition. Let $f : S^1 \to S^1$, $f(1) = z$. Then choose $u \in \Omega(S^1, z, 1)$, then $f_\star : \pi_1(S^1, 1) \to \pi_1(S^1, z)$, giving $u_\sharp \circ f_\star : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$. This is a homomorphism $\mathbb{Z} \to \mathbb{Z}$, so is uniquely determined by its action on 1. We define the *degree* of f, written $\deg f$, to be $(u_\sharp \circ f_\star)(1)$.

By the above lemma, this definition does not depend on the choice of path u.

Example. Let $\gamma_n \in \Omega(S^1, 1)$ be given by $\gamma_n(t) = e^{2\pi i n t}$ for $n \in \mathbb{Z}$. Then $\hat{\gamma}_n(t) = nt$, so $\varepsilon_p[\gamma_n] = n$. The integers n correspond to the classes $[\gamma_n]$ in $\pi_1(S^1, 1)$.

Let $f_n = \overline{\gamma}_n : S^1 \to S^1$, so $f_n(z) = z^n$. Then $f_n \circ \gamma_1 = \gamma_n$, so $f_{n\star}[\gamma_1] = [\gamma_n]$. Hence the degree of f_n is n.

Proposition. The degree of $f_n : S^1 \to S^1$, defined by $z \mapsto z^n$, is n. If $g_0, g_1 : S^1 \to S^1$, then $g_0 \sim g_1$ if and only if $\deg g_0 = \deg g_1$. $g : S^1 \to S^1$ extends to $G : D^2 \to S^1$ if and only if $\deg g = 0$.

Proof. Suppose $g_0 \sim g_1$ via $H : S^1 \times I \to S^1$. Let $u(t) = H(1, t)$, so $g_{1\star} = u_\sharp \circ g_{0\star}$, where $u \in \Omega(S^1, g_0(1), g_1(1))$. Let $v \in \Omega(S^1, g_1(1), 1)$. Then $uv \in \Omega(S^1, g_0(1), 1)$, and so $\deg g_1 = v_\sharp \circ g_{1\star}(1) = v_\sharp(u_\sharp \circ g_0(1)) = (uv)_\sharp \circ g_{0\star}(1) = \deg g_0$, since $u_\sharp[\gamma] = [u^{-1}\gamma u]$ so $(u \circ v)_\sharp = v_\sharp \circ u_\sharp$.

Conversely, it suffices to show that $g \sim f_{\deg g}$ by transitivity. Suppose $g(1) = 1$. Then $g = \overline{\gamma}$ where $\gamma = g \circ \gamma_1$. Then $\deg g = g_\star(1) = [g \circ \gamma_1] = [\gamma] \in \pi_1(S^1, 1)$. In particular, if $\deg g = n$, we have $\gamma \sim \gamma_n$, so $g = \overline{\gamma} \sim \overline{\gamma}_n = f_n$.

In general, if $g(1) = e^{2\pi i x}$, then $g \sim g_0$ where $g_0(z) = e^{-2\pi i x} g(z)$ via $g_t(z) = e^{-2\pi i t x} g(z)$. Then $g \sim g_0$ so $\deg g = \deg g_0$, so in particular $g \sim g_0 \sim \gamma_{\deg g}$.

g extends to D^2 if and only if $g \sim c_{S^1, z_0}$ for some $z_0 \in S^1$. Equivalently, $g \sim c_{S^1, 1} = f_0$, so $\deg g = 0$ by above. □

4.6 Fundamental theorem of algebra

Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial, so $p(w) = w^n + a_{n-1}w^{n-1} + \cdots + a_0 = w^n + q(w)$.

Lemma. Let $R_0 = \max\left\{1, \sum_{i=0}^{n-1} |a_i|\right\}$. Then if $|w| > R_0$, $|w^n| > |q(w)|$.

Proof. Consider
$$\frac{|q(w)|}{|w^{n-1}|} \leq \sum_{i=0}^{n-1} |a_i| |w|^{i-n+1}$$
Hence, if $|w| > 1$, each term $|w|^{i-n+1}$ is at most one.
$$\sum_{i=0}^{n-1} |a_i| |w|^{i-n+1} \leq \sum_{i=0}^{n-1} |a_i| \leq R_0$$
Hence $\frac{|q(w)|}{|w|^n} < \frac{R_0}{|w|} < 1$. □

Consider $g_0, g_1 : S^1 \to \mathbb{C} \setminus \{0\}$ given by $g_0(z) = (Rz)^n$ for some fixed $R > R_0$, and $g_1(z) = p(Rz)$. Then $g_0 \sim g_1$ via $g_t(z) = p_t(Rz)$ where $p_t(w) = w^n + tq(w)$. This map has codomain $\mathbb{C} \setminus \{0\}$ by the above lemma. Let $\pi : \mathbb{C} \setminus \{0\} \to S^1$ be the radial projection $w \mapsto \frac{w}{|w|}$. Then $\pi \circ g_0, \pi \circ g_1 : S^1 \to S^1$ are homotopic maps. Therefore, $n = \deg(\pi \circ g_0) = \deg(\pi \circ g_1)$.

Theorem. If $n > 0$, p has a root $w_0 \in \mathbb{C}$.

Proof. If $p(w) \neq 0$ for all w, $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, so g_1 extends to $G_1 : D^2 \to \mathbb{C} \setminus \{0\}$ given by $G_1(z) = p(Rz)$. Then $\pi \circ G_1$ is an extension of $\pi \circ g_1$. So $n = \deg \pi \circ g_1 = 0$, so we have a constant polynomial. □

4.7 Wedge product

Definition. Let (X_i, x_i) be pointed spaces. The *wedge product* $\bigvee_{i=1}^n (X_i, x_i) = \coprod_{i=1}^n (X_i, x_i) / \sim$ for the equivalence relation $\sim$ generated by $x_i \sim x_j$. For $n = 2$, we also write $(X_1, x_1) \vee (X_2, x_2)$ for $\bigvee_{i=1}^2 (X_i, x_i)$.

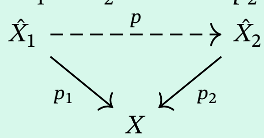
If each X_i has the property that for any $x_i, x_i' \in X_i$, there exists a homeomorphism $\varphi : X_i \to X_i$ such that $\varphi(x_i) = \varphi(x_i')$, then the particular choice of base point used in the wedge product does not matter, and the expression $\bigvee_{i=1}^n X_i = \bigvee_{i=1}^n (X_i, x_i)$ is well-defined up to homeomorphism independent of the choice of the x_i.

Example. Consider the figure-eight $S^1 \vee S^1$. There are inclusion maps $\iota_1, \iota_2 : (S^1, 1) \to (S^1 \vee S^1, x_0)$ where x_0 is the point at which the two circles are joined. Let $a = \iota_{1\star}(1) \in \pi_1(S^1 \vee S^1, x_0)$, and similarly let $b = \iota_{2\star}(1) \in \pi_1(S^1 \vee S^1, x_0)$. The universal cover of $S^1 \vee S^1$ is the infinite regular 4-valent tree, $T_\infty(4)$. If $T_n(4)$ is the regular 4-valent tree of depth n, $T_\infty(4) = \bigcup_{n=1}^\infty T_n(4)$, so $U \subseteq T_\infty(4)$ is open if and only if $U \cap T_n(4)$ is open for all n. There is a covering map from $T_\infty(4)$ to $S^1 \vee S^1$ by mapping each edge to one of the circles. $T_\infty(4)$ is simply connected, because the interval I is compact, so if $\gamma : I \to T_\infty(4)$, $\operatorname{Im} \gamma \subseteq T_n(4)$ for some n, and each of the finite trees is contractible and therefore simply connected.

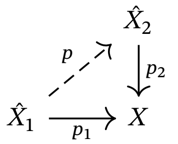
In particular, there is a bijection $\pi_1(S^1 \vee S^1, x_0) \to p^{-1}(\{x_0\})$ given by $[\gamma] \to \varepsilon_p(\gamma)$. Here, $\varepsilon_p(ab) = \widehat{ab}(1)$, but $\varepsilon_p(ba) = \widehat{ba}(1) \neq \widehat{ab}(1)$. In $\pi_1(S^1 \vee S^1, x_0)$, $ab \neq ba$, so $\pi_1(S^1 \vee S^1, x_0)$ is not abelian.

4.8 Covering transformations

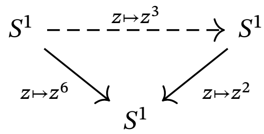
Definition. Let $p_i : \hat{X}_i \to X$ be covering maps for $i = 1, 2$. A *covering transformation* $p : (p_1, \hat{X}_1) \to (p_2, \hat{X}_2)$ is a map $p : \hat{X}_1 \to \hat{X}_2$ such that $p_2 \circ p = p_1$.



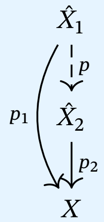
Remark. We can think of p as a lift of p_1 to $\hat{X}_2$.



Example. Let $p_1 : S^1 \to S^1$ be defined by $z \mapsto z^6$, and $p_2 : S^1 \to S^1$ be defined by $z \mapsto z^2$. Then $p : (p_1, S^1) \to (p_2, S^1)$ defined by $z \mapsto z^3$ is a covering transformation.



Lemma. Let X be locally path-connected. If $p : (p_1, \hat{X}_1) \to (p_2, \hat{X}_2)$ is a covering transformation, $p : \hat{X}_1 \to \hat{X}_2$ is a covering map.



Proof. Given $x_2 \in \hat{X}_2$, we find an open evenly covered neighbourhood U_{x_2}. Let $x = p_2(x_2) \in X$. Then p_1, p_2 are covering maps of X, so there exist open neighbourhoods U_1, U_2 of x such that U_i is evenly covered by p_i. Then $U = U_1 \cap U_2$ is open and evenly covered by p_1 and p_2. Since X is locally path-connected, let $V \subseteq U$ be an open neighbourhood of x that is path-connected. Then $p_1^{-1}(V) = \coprod_{\alpha \in A} V_\alpha$ and $p_2^{-1}(V) = \coprod_{\beta \in B} V_\beta$, where $V_\alpha \simeq V \simeq V_\beta$ are all path-connected. Let $x_\alpha = p_{1,\alpha}^{-1}(x)$, and $x_\beta = p_{2,\beta}^{-1}(x)$. Then $p_2(p(x_\alpha)) = p_1(x_\alpha) = x$, so $p(x_\alpha) = x_\beta$ for some $\beta \in B$. Now, V_α, V_β are path-connected, so $p(V_\alpha) \subseteq V_\beta$ since each V_β is a (maximal) path-connected component of $p_2^{-1}(V)$. Therefore, $p|_{V_\alpha} : V_\alpha \to V_\beta$ satisfies $p_{2,\beta} \circ p|_{V_\alpha} = p_{1,\alpha}$, so $p|_{V_\alpha} = p_{2,\beta}^{-1} \circ p_{1,\alpha}$ is a homeomorphism.

In particular, $p^{-1}(V_\beta) = \coprod_{\alpha \in V, p(x_\alpha) = x_\beta} V_\alpha$, and $p|_{V_\alpha} : V_\alpha \to V_\beta$ is a homeomorphism. So V_β is evenly covered, so p is indeed a covering map. $\square$

4.9 Uniqueness of universal covers

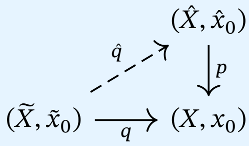
Let X be a locally path-connected space, and $q : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal cover. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$.

Lemma. If $p : \hat{Y} \to Y$ is a bijective covering map, then p is a homeomorphism.

Proof. p is continuous and bijective, therefore $p^{-1} : Y \to \hat{Y}$ exists as a map of sets. We must show that this map is continuous. Since p is a covering map, Y has an open cover $\{U_y \mid y \in Y\}$ such that U_y is evenly covered. In particular, $p^{-1}|_{U_y} : U_y \to p^{-1}(U_y)$ is a homeomorphism. Hence p^{-1} is continuous. $\square$

Recall that if $p_i : \hat{X}_i \to X$ are covering maps, a covering transformation from $(p_1, \hat{X}_1)$ to $(p_2, \hat{X}_2)$ is a lift $\hat{p}_1$ of p_1 to X_2. $\hat{p}_1$ is a covering isomorphism if it is bijective. Then, by the lemma, it is a homeomorphism.

Proposition. Let X be a locally path-connected space, and $q : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal cover. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$. Then there is a unique covering transformation $\hat{q} : (p, \hat{X}) \to (q, \widetilde{X})$



Proof. Note that $\widetilde{X}$ is simply connected, and since X is locally path-connected, so is $\widetilde{X}$. So existence and uniqueness of $\hat{q}$ is exactly the simply connected lifting property. $\square$

Corollary. If p is also a universal cover, $\hat{q}$ is a covering isomorphism, and in particular, $\hat{X} \simeq \widetilde{X}$.

Proof. $\widetilde{X}$ is simply connected, so $\hat{q} : \widetilde{X} \to \hat{X}$ is a universal cover. Hence, there is a bijection between points $\hat{q}^{-1}(\hat{x})$ and elements $\pi_1(\hat{X}, \hat{x})$. But this is the one-element set, since $\hat{X}$ is simply connected. So $\hat{q}^{-1}(\hat{x})$ has a single element, and so $\hat{q}$ is a bijection. $\square$

Equivalently, if $q : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ and $q' : (\widetilde{X}', \tilde{x}_0') \to (X, x_0)$ are universal covers, there is a unique covering isomorphism $\hat{q} : (\widetilde{X}, \tilde{x}_0) \to (\widetilde{X}', \tilde{x}_0')$.

4.10 Deck groups

Definition. The *deck group* $G_D(p)$ is the set of covering automorphisms $g : (p, \hat{X}) \to (p, \hat{X})$, which forms a group under composition $gf = g \circ f$. This has a left action on $\hat{X}$ by $g \cdot \hat{x} = g(\hat{x})$.

Example. Let $p : (\mathbb{R}, 0) \to (S^1, 1)$. The deck group $G_D(p)$ is exactly

$$\{g_n : \mathbb{R} \to \mathbb{R} \mid g_n(t) = t + n\} \simeq \mathbb{Z}$$

In this case, $G_D(p) \simeq \pi_1(S^1, 1)$.

Example. There is a bijection between $G_D(q)$ and $q^{-1}(x_0)$, by $g \mapsto g(\tilde{x}_0)$, by the above proposition with $\hat{X} = \widetilde{X}$.

Theorem. Let $q : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal cover. Then $G_D(q) \simeq \pi_1(X, x_0)$.

Proof. There is a bijection between $\pi_1(X, x_0)$ and $q^{-1}(x_0)$ since q is a universal cover. By the above example, $q^{-1}(x_0)$ is in bijection with $G_D(q)$. In particular, we can map $[\gamma] \in \pi_1(X, x_0)$ to $\tilde{\gamma}(1) \in q^{-1}(x_0)$, where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_0$, and $g(\tilde{x}_0) \in q^{-1}(x_0)$ is mapped to $g \in G_D(q)$. We need to check that this composed map is a homomorphism: it is already a bijection of sets.

$[\gamma\gamma']$ is mapped to $\widetilde{\gamma\gamma'}(1) = \tilde{\gamma}(g_{\tilde{\gamma}(1)} \circ \tilde{\gamma})$ where $g_{\tilde{\gamma}(1)}$ is the unique element of $G_D(q)$ with $g_{\tilde{\gamma}(1)}(x_0) = \tilde{\gamma}(1)$. Since $g_{\tilde{\gamma}(1)} \circ \tilde{\gamma}'$ is a lift of $\tilde{\gamma}'$ starting at $\tilde{\gamma}(1)$, we have $\widetilde{\gamma\gamma'}(1) = (g_{\tilde{\gamma}(1)} \circ \tilde{\gamma}')(1) = g_{\tilde{\gamma}(1)}(\tilde{\gamma}'(1)) = g_{\tilde{\gamma}(1)}(g_{\tilde{\gamma}'(1)}(\tilde{x}_0))$. So $\widetilde{\gamma\gamma'}(1)$ is the image of $\tilde{x}_0$ under $g_{\tilde{\gamma}(1)} \circ g_{\tilde{\gamma}'(1)}$, so this is indeed a homomorphism. □

4.11 Correspondence of subgroups and covers

Proposition. Let $G = G_D(q) \simeq \pi_1(X, x_0)$. If $H \leq G$ is a subgroup, we have a tower of covering maps

$$\begin{array}{ccc} \widetilde{X} & & 1 \\ \downarrow{\scriptstyle \pi_H} & & \uparrow \\ X_H & & H \\ \downarrow{\scriptstyle p_H} & & \uparrow \\ X & & G \end{array}$$

where $X_H = H \backslash \widetilde{X}$ is the quotient given by $h \cdot x \sim x$ for all $h \in H$. In particular, $\pi_H : \widetilde{X} \to H \backslash \widetilde{X}$ is the quotient map, and $p_H : X_H \to X$ is given by $p_H(H \cdot x) = q(x)$. This is well-defined because $q \circ h = q$ as h is a deck transformation. In particular, if $H = G$, p_G is a covering isomorphism, so $X \simeq G \setminus \widetilde{X}$.

A universal covering map is a quotient by the action of $G_D(q) \simeq \pi_1(X, x_0)$.

Proof. Let $x \in X$. Then choose U_x to be evenly covered by q. Then $q^{-1}(U_x) = \coprod_{\alpha \in A} U_\alpha = \coprod_{g \in G_D(q)} g \cdot U_{\alpha_0}$ for $\tilde{x}_0 \in U_{\alpha_0}$. Then $p_H^{-1}(U_x) = \coprod_{\beta = gH \in \text{cosets of } H} U_\beta$. Then $\pi_H^{-1}(U_\beta) = \coprod_{gh \in gH} gh \cdot U_{\alpha_0}$, and $p_H^{-1}(U_x) = \coprod U_\beta$. So each is evenly covered. □

Definition. $p : \widetilde{X} \to X$ is a *normal cover* if $G_D(p)$ acts transitively on $p^{-1}(x_0)$.

Example. The universal cover q is always a normal cover.

Proposition. Let $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ be a covering map. Then $p_\star : \pi_1(\hat{X}, \hat{x}_0) \to \pi_1(X, x_0)$ is injective. In particular, $\operatorname{Im} p_\star \simeq \pi_1(\hat{X}, \hat{x}_0)$ is a subgroup of $\pi_1(X, x_0)$.

Proof. If $p_\star[\gamma_0] = p_\star[\gamma_1]$, we have $p \circ \gamma_0 \sim_e p \circ \gamma_1$, so $p \circ \hat{\gamma}_0 \sim_e p \circ \hat{\gamma}_1$, so $\gamma_0 \sim_e \gamma_1$. In particular, $[\gamma_0] = [\gamma_1]$. □

Let $q : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal cover, so $\widetilde{X}$ and hence X are path-connected. Suppose further that X is locally path-connected, so $\widetilde{X}$ is also locally path-connected. Consider

$$S(X, x_0) = \{H \leq \pi_1(X, x_0)\}$$
$$C(X, x_0) = \left\{(p, \hat{X}, \hat{x}_0) \mid p : (\hat{X}, \hat{x}_0) \to (X, x_0) \text{ is a covering map, } \hat{X} \text{ is path-connected}\right\} \big/ \sim$$

where $(p, \hat{X}, \hat{x}_0) \sim (p', \hat{X}', \hat{x}_0')$ if there is a covering isomorphism $q : (p, \hat{X}) \to (p', \hat{X}')$ mapping $\hat{x}_0 \mapsto \hat{x}_0'$. Let $\alpha : S(X, x_0) \to C(X, x_0)$ be given by $\alpha(H) = (p_H, X_H, x_{0,H})$, where $X_H = H \setminus \widetilde{X}$, so $\widetilde{X} \xrightarrow{\pi_H} X_H \xrightarrow{p_H} X$ mapping $\tilde{x}_0$ to $x_{0,H}$. Let $\beta : C(X, x_0) \to S(X, x_0)$ be defined by $(p, \hat{X}, \hat{x}_0) \mapsto p_\star(\pi_1(\hat{X}, \hat{x}_0))$.

Theorem. α, β are inverses, and hence bijections.

Remark. The entire group $G = \pi_1(X, x_0)$ is mapped to (id, X, x_0). The trivial group $1 \subsetneq G$ is mapped to the universal cover $(q, \widetilde{X}, \tilde{x}_0)$. The index $[G : H]$ is exactly $|p_H^{-1}(x_0)|$. A conjugation $g^{-1}Hg$ corresponds to a change of base point $(p_H, X_H, \hat{\gamma}(1))$, where $g = [\gamma]$ and $\hat{\gamma} : I \to X_H$ is a lift of γ with $\hat{\gamma}(0) = x_{0,H}$. If $H \trianglelefteq G$ is a normal subgroup, p_H is a normal covering. The quotient ${G}/{H}$ corresponds to the deck group $G_D(p_H)$.

Proof. Consider $\beta(\alpha(H)) = p_{H\star}(\pi_1(X_H, x_{0,H}))$. There are isomorphisms

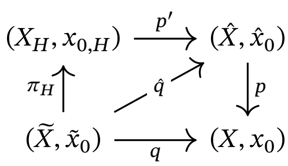
$$H \to \pi_1(X_H, x_{0,H}) \to p_{H\star}(\pi_1(X, x_0))$$

mapping

$$[\gamma] \mapsto [\pi_H \circ \tilde{\gamma}] \mapsto [p_H \circ \pi_H \circ \tilde{\gamma}] = [\pi_G \circ \tilde{\gamma}] = [\gamma]$$

where $\tilde{\gamma}$ is a lift of γ such that $\tilde{\gamma}(0) = \tilde{x}_0$. Hence $\beta(\alpha(H)) = H$.

Conversely, consider $\alpha(\beta((p, \hat{X}, \hat{x}_0))) = (p_H, X_H, x_{0,H})$ where $H = p_\star(\pi_1(X, x_0))$. Consider



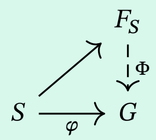
We claim that $\hat{q} = p' \circ \pi_H$, where p' is a covering isomorphism. If we can show this, we have $(p_H, X_H, x_{0,H}) \sim (p, \hat{X}, \hat{x}_0)$, so $\alpha \circ \beta$ is the identity on $C(X, x_0)$. If $h \in H = p_\star(\pi_1(\hat{X}, \hat{x}_0))$, $h = [p \circ \gamma]$ for some $\gamma \in \Omega(\hat{X}, \hat{x}_0)$. Then $\hat{q}(\tilde{x}) = \widehat{q \circ \eta_{\tilde{x}}}(1)$ where $\eta_{\tilde{x}} \in \Omega(\widetilde{X}, \tilde{x}_0, \tilde{x})$. Then $\eta_{h\cdot\tilde{x}} = \eta_{h \circ \tilde{x}_0}(h \circ \eta_{\tilde{x}})$, so $q \circ \eta_{h\cdot\tilde{x}} = (q \circ \eta_{h\cdot\tilde{x}_0})(q \circ \eta_{\tilde{x}}) = (p \circ \gamma)(q \circ \eta_{\tilde{x}})$, so in particular, $\widehat{q \circ \eta_{h\cdot\tilde{x}}} = (\gamma)(\widehat{q \circ \eta_{\tilde{x}}})$. Hence $\hat{q}(h \cdot \tilde{x}) = (q \circ \eta_{h\cdot\tilde{x}})(1) = \widehat{q \circ \eta_{\tilde{x}}}(1) = \hat{q}(\tilde{x})$, so $\hat{q}$ factors as shown. $\hat{X}$ is connected, so p' is surjective, so it is bijective and hence a covering isomorphism. □

5 Seifert–Van Kampen theorem

5.1 Free groups and presentations

Consider $\pi_1(S^1 \vee S^1, x_0)$ where x_0 is the wedge point. The universal cover is the infinite 4-valent tree $T_\infty(4)$, so $\pi_1(S^1 \vee S^1)$ is in bijection with $q^{-1}(x_0)$, the vertices of $T_\infty(4)$. Let $\tilde{x}_0$ be one such vertex. If $\tilde{x}$ is a vertex, there is a unique shortest path from $\tilde{x}_0$ to $\tilde{x}$. This gives an 'address' for $\tilde{x}$ in $T_\infty(4)$ given by recording the type and direction of each edge used in the path. The set of such 'addresses' is in bijection with the set of *reduced words* $w = \ell_1 \dots \ell_r$ where $r \in \mathbb{N}$, and each l_i is one of a, a^{-1}, b, b^{-1}, such that w does not contain any substring of the form $aa^{-1}, a^{-1}a, bb^{-1}b^{-1}b$. Then each word w corresponds to an element $w \in \pi_1(S^1 \vee S^1, x_0)$, the image of the shortest path under q. Note that the multiplication ww' in $\pi_1(S^1 \vee S^1, x_0)$ corresponds to concatenation of words ww' and then the reduction of substrings such as aa^{-1}.

Definition. A *free group* with generating set S is a group F_S and a subset $S \subseteq F_s$ such that if G is a group and $\varphi : S \to G$ is a map of sets, there is a unique homomorphism $\Phi : F_s \to G$ with $\Phi|_S = \varphi$.



Remark. The action of taking the free group of a set is a functor from **Set** to **Grp**, and it is left adjoint to the forgetful functor from **Grp** to **Set**. This property is known as the universal property of the free group.

Example. $\pi_1(S^1 \vee S^1) \simeq F_{\{a,b\}}$. Indeed, given $\varphi : \{a, b\} \to G$, we define $\Phi(\ell_1 \dots \ell_r) = \varphi(\ell_1) \dots \varphi(\ell_r)$, where we extend φ to all of $\{a, a^{-1}, b, b^{-1}\}$ by defining $\varphi(a^{-1}) = \varphi(a)^{-1}$ and $\varphi(b^{-1}) = \varphi(b)^{-1}$. This is a homomorphism: indeed,

$$\Phi(ww') = \varphi(\ell_1) \dots \varphi(\ell_k)\varphi(\ell'_1) \dots \varphi(\ell'_k) = \Phi(w)\Phi(w')$$

cancelling substrings of the form aa^{-1} as required. The homomorphism is unique as required for the universal property of the free group.

Lemma. Let F_S, F_T be free groups on sets $S \subseteq F_S, T \subseteq F_T$. Let $\varphi : S \to T$ be a bijection. Then $\Phi : F_S \to F_T$ is an isomorphism.

Proof. Let $\psi = \varphi^{-1}$. Since F_T is free, there exists a homomorphism $\Psi : F_T \to F_S$ such that $\Psi|_T = \psi$. Then $\Psi \circ \Phi : F_S \to F_S$ has the property that for all $s \in S$, we have $\psi \circ \varphi(s) = s$. F_S is free, so there is a unique homomorphism $\alpha : F_S \to F_S$ mapping $s \in S$ to s. So $\alpha = \mathrm{id}_{F_S}$. Hence $\Psi \circ \Phi = \mathrm{id}_{F_S}$, so by symmetry, they are inverse functions. □

Corollary. If F_S, F'_S are free groups generated by S, $F_S \simeq F'_S$. So the isomorphism type of F_S depends only on $|S|$, the cardinality of S.

We therefore can write F_n for *the* free group (up to isomorphism) generated by n elements $a_1, \dots, a_n$. Let $X = \bigvee_{i=1}^n S^1$ where x_0 is the wedge point, with inclusion maps $j_n : S^1 \to X$. Let $a_i = j_{i\star}(1)$ for $1 \in \pi_1(S^1, 1)$ be a generator. Then X has universal cover $\widetilde{X} = T_\infty(2n)$, the infinite regular $2n$-valent tree. In particular, $\pi_1(X, x_0)$ is the set of reduced words in $\{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$, which is isomorphic to F_{2n}.

5.2 Presentations

Definition. Let G be a group and $S \subseteq G$ be a subset. Let $\mathcal{S}_S = \{H \leq G \mid S \subseteq H\}$, then let $\langle S \rangle = \bigcap_{H \in \mathcal{S}_S} H$ be the smallest subgroup of G containing S, known as the *subgroup generated by S*. Similarly, let $\mathcal{N}_S = \{N \trianglelefteq G \mid S \subseteq H\}$, and let $\langle\langle S \rangle\rangle = \bigcap_{H \in \mathcal{N}_S} H$ be the smallest normal subgroup of G containing S, called the *subgroup normally generated by S*.

Note that $\langle S \rangle$ is nonempty since $1 \in H$ for all $H \in \mathcal{S}_S$.

If $\langle S \rangle = G$, we say that S *generates* G. If so, there is a unique homomorphism $\Phi_S : F_S \to G$ that maps s to s. $\operatorname{Im} \Phi_S \leq G$, and it contains S, so Φ_S is surjective.

Definition. Given a set S and $R \subseteq F_S$, we define $\langle S \mid R \rangle = {F_S}/{\langle\langle R \rangle\rangle}$. If in addition $\langle\langle R \rangle\rangle = \ker \Phi_S$, then $G \simeq {F_S}/{\ker \Phi_S} = {F_S}/{\langle\langle R \rangle\rangle}$. We say $\langle S \mid R \rangle$ is a *presentation* for G.

Proposition. Any group G admits a presentation.

Proof. Clearly $\langle G \rangle = G$, so let $S = G$. Let $R = \ker \Phi_G$, where $\Phi_G : F_G \to G$. Then by construction, ${F_S}/{\langle\langle R \rangle\rangle} = {F_S}/{\ker \Phi_G} \simeq G$. □

Remark. These presentations are very large. It is often more useful to consider *finite* presentations of G, where both S and R are finite.

Example. $\langle a, b \mid \rangle \simeq F_2$. $\langle a \mid \rangle \simeq F_1 = \pi_1(S^1, 1) \simeq \mathbb{Z}$. $\langle a \mid a^3 \rangle \simeq {\mathbb{Z}}/{3\mathbb{Z}}$. $\langle a, b \mid ab^{-3} \rangle \simeq \mathbb{Z}$.

Proposition. Let $\langle S \mid R \rangle$ be a presentation, let $a \notin S$, and let $w \in F_S$. Then $\langle S \mid R \rangle \simeq \langle S \cup \{a\} \mid R \cup \{aw^{-1}\} \rangle$.

Proof. We have homomorphisms $\varphi : \langle S \mid R \rangle \to \langle S \cup \{a\} \mid R \cup \{aw^{-1}\} \rangle$ mapping $s \in S$ to s, and $\psi : \langle S \cup \{a\} \mid R \cup \{aw^{-1}\} \rangle \to \langle S \mid R \rangle$ mapping $s \in S$ to s and a to w. These are inverses. □

There are other operations we can apply to presentations. If $w \in R$, we can replace w with a conjugate sws^{-1} for $s \in S$, and it leaves the group unchanged. For example, $\langle ab \mid abb \rangle = \langle ab \mid bab \rangle$. Also, if $w_1, w_2 \in R$, we can replace w_1 with $w_1 w_2$, so for example,

$$\langle ab \mid babb, abb \rangle = \langle ab \mid b, abb \rangle \simeq \langle a \mid a \rangle \simeq 1$$

Theorem. Given a finite set S and a finite set of relations $R \subseteq F_S$, there is no algorithm to determine if $\langle S \mid R \rangle \simeq 1$.

5.3 Covering with a pair of open sets

Theorem. Let $U_1, U_2 \subseteq X$ be open, and $U_1 \cap U_2$ be path-connected with $x_0 \in U_1 \cap U_2$ and $U_1 \cup U_2 = X$. Then $\iota_{1\star}(\pi_1(U_1, x_0)) \cup \iota_{2\star}(\pi_1(U_2, x_0))$ generates $\pi_1(X, x_0)$, where $\iota_i : U_i \to X$ is the inclusion.

Proof. $\{U_1, U_2\}$ is an open cover of X, so if $\gamma \in \Omega(X, x_0)$, we have $\{\gamma^{-1}(U_1), \gamma^{-1}(U_2)\}$ is an open cover of I. By the Lebesgue covering lemma, we can find $n \in \mathbb{N}$ such that $\left[\frac{j}{n}, \frac{j+1}{n}\right]$ lies entirely inside $\gamma^{-1}(U_1)$ or $\gamma^{-1}(U_2)$ for all j. Each interval $\left[\frac{j}{n}, \frac{j+1}{n}\right]$ with the label 1 or 2 accordingly; if it lies in both, choose an arbitrary label. Let $0 = t_0 < t_1 < \dots < t_k = 1$ be the points of the form $\frac{j}{n}$ where the labelling changes. Let $I_i = [t_{i-1}, t_i]$ for each $i \in \{0, \dots, k\}$. Let $\gamma_i = \gamma|_{I_i}$, so $\gamma(t_i) \in U_1 \cap U_2$, and $\gamma(I_i) \subseteq U_{i \bmod 2}$ without loss of generality. Note that we can write γ as the composition of paths $\gamma = \gamma_1 \dots \gamma_k$.

Let $\eta_1, \dots, \eta_{k-1}$ be paths with $\eta_i \in \Omega(U_1 \cap U_2, \gamma(t_i), x_0)$, which exists since $U_1 \cap U_2$ is path-connected. Then

$$\gamma \sim_e \gamma_1 \eta_1 \eta_1^{-1} \gamma_2 \eta_2 \eta_2^{-1} \dots \eta_{k-1} \eta_{k-1}^{-1} \gamma_k = \underbrace{(\gamma_1 \eta_1)}_{\delta_1} \underbrace{(\eta_1^{-1} \gamma_2 \eta_2)}_{\delta_2} \eta_2^{-1} \dots \eta_{k-1} \underbrace{(\eta_{k-1}^{-1} \gamma_k)}_{\delta_k}$$

Then each $\delta_i \in \Omega_i(U_1, x_0)$, so $[\delta_i] \in \operatorname{Im} \iota_{(i \bmod 2)\star}$. So $[\gamma] = [\delta_1][\delta_2] \dots [\delta_k]$ is a product of elements in $\iota_{1\star}(\pi_1(U_1, x_0)) \cup \iota_{2\star}(\pi_1(U_2, x_0))$, so $[\gamma]$ lies in the subgroup they generate. ☐

Corollary. Let $U_1, U_2 \subseteq X$ be open and simply connected with $U_1 \cup U_2 = X$, and let $U_1 \cap U_2$ be path-connected and contain x_0. Then X is simply connected.

Proof. $\pi_1(X, x_0)$ is generated by $\iota_{1\star}(\pi_1(U_1, x_0)) \cup \iota_{2\star}(\pi_1(U_2, x_0)) = \{1\}$. ☐

Example. $S^n = U^+ \cup U^-$, where $U^+ = S^n = \{(1, 0, \dots, 0)\}$ and $U^- = S^n - \{(-1, 0, \dots, 0)\}$. Then $U^+ \simeq U^- \simeq \mathbb{R}^n$ by stereographic projection. $U^+ \cap U^- \simeq \mathbb{R}^n - \{0\}$. Hence $\pi_1(U^\pm, x_0) = 1$ since $\mathbb{R}^n$ is contractible. $U^+ \cap U^-$ is path-connected if $n > 1$, so $\pi_1(S^n, x_0) = 1$ for $n > 1$.

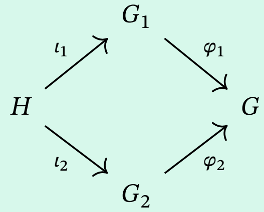
Example (attaching a disk). If $f : S^1 \to X$ with $f(1) = x_0$, let $X \cup_f D^2 = {X \amalg D^2}/{\sim}$, where $\sim$ is the smallest equivalence relation such that $z \sim f(z)$ for $z \in S^1$. Let π be the quotient map from $X \amalg D^2$ to $X \cup_f D^2$. Then let $U_1 = \pi(X \cup D^2 \setminus \{0\})$ and $U_2 = \pi(D^2)$. Then $U_1 \cup U_2 = X \cup_f D^2$, and $U_1 \cap U_2 = (D^2)^\circ \setminus \{0\}$ is path-connected. $\pi_1(U_2) = 1$, so $\pi_1(X \cup_f D^2)$ is generated by $\pi_1(X)$. Note that $f_\star : \pi_1(S^1, 1) \to \pi_1(X, x_0)$, so $f_\star(1)$ lies in the kernel of the inclusion $\pi_1(X, x_0) \to \pi_1(X \cup_f D^2, x_0)$, since $f_\star(1)$ is null-homotopic in $X \cup_f D^2$. So $\pi_1(X \cup_f D^2)$ surjects onto ${\pi_1(X)}/{\langle\langle f_\star(1) \rangle\rangle}$.

This is in fact an isomorphism. Suppose $[\gamma] \in \pi_1(X \cup_f D^2, x_0)$ is mapped to the trivial element of $\pi_1(X)/\langle\langle f_\star(1)\rangle\rangle$, so $[\gamma]$ can be viewed as an element of $\langle\langle f_\star(1)\rangle\rangle$. Note that all such $[\gamma]$ are of the form $a_1 f_\star(n_1)a_1^{-1} \dots a_k f_\star(n_k)a_k^{-1}$. Since $f_\star(n) = 1$ in $\pi_1(X \cup_f D^2, x_0)$, $[\gamma] = 1$.

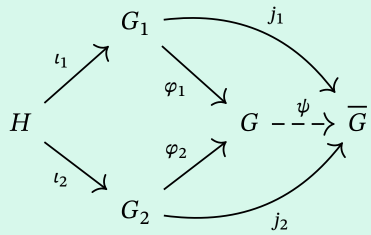
5.4 Amalgamated free products

Definition. Let $\iota_1 : H \to G_1, \iota_2 : H \to G_2$ be group homomorphisms. A group G is an *amalgamated free product of G_1 and G_2 along H* if:

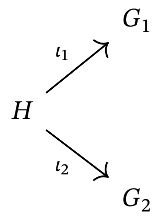
(i) There are homomorphisms $\varphi_1 : G_1 \to G, \varphi_2 : G_2 \to G$ such that the following diagram commutes.



(ii) It is universal with this property, so for any other group $\overline{G}$ with a commutative square as above, there is a unique homomorphism $\psi : G \to \overline{G}$ such that the following diagram commutes.



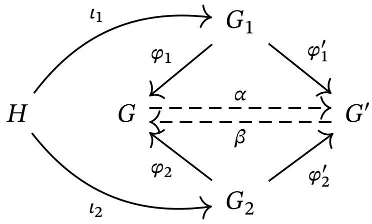
Remark. The amalgamated free product is the colimit of the following diagram.



Hence, it is a categorical pushout.

Proposition. If G, G' are amalgamated products of G_1, G_2, then $G \simeq G'$.

Proof. There are homomorphisms $\alpha : G \to G', \beta : G' \to G$, and the uniqueness in the definition implies $\alpha \circ \beta = \mathrm{id}_{G'}$ and $\beta \circ \alpha = \mathrm{id}_G$. In other words, the following diagram commutes.



□

Proposition. An amalgamated product of any two groups exists.

The universal property of the presentation is that $\langle S \mid R \rangle \simeq {}^{F_S}\!/\!_{\langle\langle R \rangle\rangle}$. Suppose $S \subseteq G$ satisfies the relations R in G, so all of the relations map to the identity. Then there is a unique homomorphism $\langle S \mid R \rangle \to G$ mapping $s \in S$ to s, since there is a unique homomorphism $F_S \to G$ mapping $s \in S$ to s, and since S satisfies the relations, this factors through ${}^{F_S}\!/\!_{\langle\langle R \rangle\rangle}$.

For example, consider a map $\langle a \mid a^4 \rangle \to {}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$ that maps a to 1. We can check that the relation $1^4 = 0$ in ${}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$ holds.

Proof. Consider presentations $G_i = \langle S_i \mid R_i \rangle$ of G_1, G_2, and $H = \langle T \mid W \rangle$. Then define

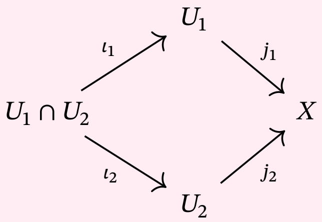
$$G = G_1 *_H G_2 = \left\langle S_1 \cup S_2 \cup T \mid R_1 \cup R_2 \cup \left\{ t_i^{-1}\iota_1(t_i), t_i^{-1}\iota_2(t_i) \mid t_i \in T \right\} \right\rangle$$

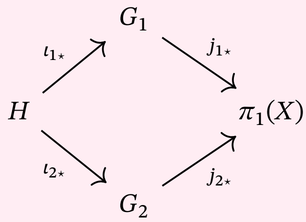
Then $\varphi_i : G_i \to G$ are given by $s \in S_i$ mapping to s. Given $j_1, j_2 : G_1, G_2 \to \overline{G}$, we define $\psi : G \to \overline{G}$ mapping $s \in S_1$ to $j_1(s)$, $s \in S_2$ to $j_2(s)$, and $t \in T$ to $j_1 \circ \iota_1(t) = j_2 \circ \iota_2(t)$, and check that the relations hold. □

This is isomorphic to $\left\langle S_1 \cup S_2 \,|\, R_1 \cup R_2 \cup \left\{ \iota_1(t_i)\iota_2^{-1}(t_i) \mid t_i \in T \right\} \right\rangle$.

5.5 Seifert–Van Kampen theorem

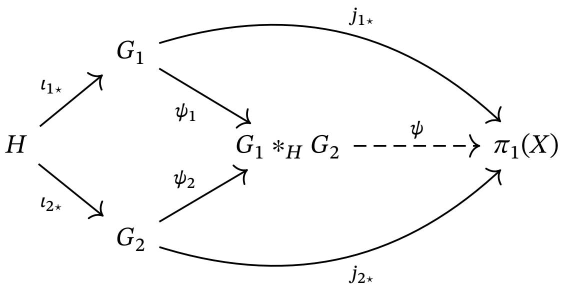
Theorem (Seifert–Van Kampen). Let $X = U_1 \cup U_2$ where U_i are open sets with $U_1 \cap U_2$ path-connected and containing x_0. Let $G_i = \pi_1(U_i, x_0)$, and $H = \pi_1(U_1 \cap U_2, x_0)$, so





Then $\pi_1(X, x_0) = G_1 *_H G_2$.

Remark. The ‘easy’ part of the proof is that we have a commutative diagram



so we obtain a map $\psi\colon G_1 *_H G_2 \to \pi_1(X, x_0)$ by universality of the amalgamated free product. Clearly ψ is surjective by the theorem in the previous subsection, and the difficult part of the proof is showing that ψ is injective.

Proof sketch. We show that if $H\colon I \times I \to X$ is a homotopy between γ_0 and γ_1, then $[\gamma_0] = [\gamma_1]$ using the relations in $G_1 *_H G_2$. We can divide $I \times I$ into squares of size $\frac{1}{n}$ such that the image of each square under H lies in either U_1 or U_2 by the Lebesgue covering lemma. Each row represents a path $\gamma_{\frac{i}{n}}$, and by operating row-by-row we will show $\gamma_{\frac{i}{n}}$ is related to $\gamma_{\frac{i+1}{n}}$ in $G_1 *_H G_2$. To move from one row to the next, if there are different labels above and below, the boundary lies in $U_1 \cap U_2$, so we use the relations $\iota_{1\star}(t_1) = \iota_{2\star}(t_1)$. □

Example. Consider $X \cup_f D^2 = U_1 \cup U_2$ where $U_1 = X \cup_f D^2 \setminus \{0\}$ and $U_2 = (D^2)^\circ$, with $x_0 \in U_1 \cap U_2$. Let $p\colon U_1 \to X$ be the inclusion. Since $D^2 \setminus \{0\}$ has a strong deformation retraction to S^1, we know U_1 has a strong deformation retraction to X, so $\pi_1(U_1, x_0) \simeq \pi_1(X, p(x_0))$. Note that $\pi_1(U_2, x_0)$ is the trivial group, since $(D^2)^\circ$ is contractible. Note that $U_1 \cap U_2 = (D^2)^\circ \setminus \{0\}$ is homotopy equivalent to S^1, so $\pi_1(U_1 \cap U_2, x_0) = \mathbb{Z} = \langle \gamma \rangle$.

Then, by the Seifert–Van Kampen theorem, we have $\pi_1(X \cup_f D^2) \simeq \pi_1(X) *_{\mathbb{Z}} 1$. If $\pi_1(X, x_0) = \langle S \mid R \rangle$, we have in particular that

$$\pi_1(X \cup_f D^2) \simeq \left\langle S, t \,\middle|\, R \cup \left\{t, t^{-1} f_\star(t)\right\} \right\rangle = \langle S \mid R \cup f_\star(t) \rangle = {\pi_1(X, x_0)}\big/{\langle\!\langle f_\star(t) \rangle\!\rangle}$$

Example. Consider the torus $T^2 = S^1 \vee S^1 \cup_f D^2$. Let a, b be generators for $\pi_1(S^1 \vee S^1)$. Then the commutator $aba^{-1}b^{-1}$ represents the disk attached. So $\pi_1(T^2) = \left\langle a, b \mid aba^{-1}b^{-1} \right\rangle = \mathbb{Z}^2$.

Example. Let Σ_g be a surface of genus g. Then $\Sigma_g = \bigvee_{i=1}^{g} (S^1 \vee S^1) \cup_f D^2$, so

$$\pi_1(\Sigma_g) \simeq \left\langle a_1, b_1, \dots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right\rangle$$

Example. A surface of genus two can be realised as a union of U_1, U_2 where $U_1 \cap U_2 \simeq S^1$ and $\pi_1(U_i) = \langle a_i, b_i \rangle$, then $\pi_1(\Sigma_2) = \langle a_1, b_1 \rangle *_{\mathbb{Z}} \langle a_2, b_2 \rangle$.

6 Simplicial complexes

6.1 Simplices

We have shown that $\pi_1(S^1, x_0) \simeq \mathbb{Z}$, and $\pi_1(S^n, x_0) \simeq 1$ for $n > 1$, so $S^1 \nsim S^n$. We would like to show that $S^n \sim S^m$ only holds if $n = m$. One proof of this fact is that any $f\colon S^n \to S^m$ with $n < m$ is null-homotopic, but the identity on S^m is not. Both of these claims require proof: simplicial complexes will allow us to prove the first, and homology will allow us to prove the second.

Definition. The *n-simplex* is the topological space

$$\Delta^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum_{i=0}^{n} x_i = 1\right\}$$

with the subspace topology.

Remark. Δ^1 is homeomorphic to I. Δ^2 is an equilateral triangle, and Δ^3 is a regular tetrahedron. For all n, Δ^n is closed and bounded in $\mathbb{R}^{n+1}$, and hence compact and Hausdorff. The standard basis vectors $e_0, \dots, e_n$ are the vertices of Δ^n.

Definition. If $I \subseteq \{0, \dots, n\}$, the *Ith face of* Δ^n is

$$e_I = \{x \in \Delta^n \mid x_i = 0 \text{ for } i \notin I\}$$

We define $F(\Delta^n) = \{e_I \mid I \subseteq \{0, \dots, n\}\}$ to be the set of faces of Δ^n.

If $I = \{i_0, \dots, i_k\}$ with $i_0 < \dots < i_k$, we write $I = i_0 i_1 \dots i_k$.

Remark. Note that $e_{\{i\}} = e_i$, and $\Delta^n = e_{\{0,1,\dots,n\}}$. e_I is a closed subset of Δ^n, and is homeomorphic to $\Delta^{|I|-1}$. $e_I \subseteq e_J$ if and only if $I \subseteq J$. $e_I \cap e_J = e_{I \cap J}$.

Definition. A map $|f| : \Delta^n \to \mathbb{R}^N$ is *affine linear* if it is the restriction of a linear map $\mathbb{R}^{n+1} \to \mathbb{R}^n$. Equivalently, $|f|\left(\sum_{i=0}^n x_i e_i\right) = \sum_{i=0}^n x_i |f|(e_i)$. We say an affine linear map $|f| : \Delta^n \to \Delta^m$ is *simplicial* if it maps vertices in Δ^n to vertices in Δ^m, so there is a map of sets $\hat{f} : \{0, \dots, n\} \to \{0, \dots, m\}$ where $|f|(e_i) = e_{\hat{f}(i)}$.

Remark. Affine linear maps are continuous, and are determined entirely by their action on e_i. In particular, simplicial maps $|f|$ are determined by $\hat{f}$. For $I \subseteq \{0, \dots, n\}$, we have $|f|(e_I) = e_{\hat{f}(I)}$.

Definition. Vectors $v_0, \dots, v_n \in \mathbb{R}^N$ are *affine linearly independent* if whenever $\sum t_i v_i = 0$ and $\sum t_i = 0$, we have $t_i = 0$ for all i. Equivalently,

(i) If $\sum t_i v_i = \sum t_i' v_i$ and $\sum t_i = \sum t_i'$, then for each i, $t_i = t_i'$.
(ii) The vectors $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent.
(iii) The unique affine linear map $|f| : \Delta^n \to \mathbb{R}^N$ given by $|f|(e_i) = v_i$ is injective.

If $v_0, \dots, v_n$ are affine linearly independent, we write

$$[v_0, \dots, v_n] = \operatorname{Im} |f| = \left\{\sum x_i v_i \mid \sum x_i = 1, x_i \geq 0\right\}$$

and we say $[v_0, \dots, v_n]$ is a *Euclidean simplex.*

Remark. Δ^n is compact and $[v_0, \dots, v_n]$ is Hausdorff, so by the topological inverse function theorem, $|f| : \Delta^n \to [v_0, \dots, v_n]$ is a homeomorphism if the v_i are affine linearly independent.

Lemma. If $X \subseteq \mathbb{R}^N$, let $Z(X)$ be the set of $x \in X$ such that if $x = \sum t_i x_i$ for $t_i > 0, \sum t_i = 1$ and all $x_i \in X$, then $x_i = x$ for some i. Then $Z([v_0, \dots, v_n]) = \{v_0, \dots, v_n\}$.

Proof. We show that $v_k \in Z([v_0, \dots, v_n])$; the converse is clear from the definition of the simplex. Suppose $v_k = \sum t_i x_i$ for $t_i > 0$ and $\sum t_i = 1$. Then $x_i = \sum_{j=0}^n s_{ij} v_j$, since $x_i \in [v_0, \dots, v_n]$. So $v_k = \sum_j \left(\sum_i t_i s_{ij}\right) v_j$. Since the v_i are affine linearly independent, and $\sum_j \left(\sum_i t_i s_{ij}\right) = 1$, we must have $\sum t_i s_{ij} = 0$ for $j \neq k$. But $t_i > 0$ and $s_{ij} \geq 0$, so the only case is when all s_{ij} are exactly zero for $j \neq k$, so $x_j = v_k$. □

Corollary. If $[v_0, \dots, v_n] = [v_0', \dots, v_n']$ as subsets of $\mathbb{R}^N$, then $\{v_0, \dots, v_n\} = \{v_0', \dots, v_n'\}$ as sets.

Therefore, a simplex determines its set of vertices.

Proof. $\{v_0, \dots, v_n\} = Z([v_0, \dots, v_n]) = Z([v_0', \dots, v_n']) = \{v_0', \dots, v_n'\}$. □

Definition. $\mathcal{S}(\mathbb{R}^n)$ is the set of Euclidean simplices $\sigma \subseteq \mathbb{R}^n$. Hence, $\mathcal{S}(\mathbb{R}^n)$ is in bijection with the set $\left\{\{v_0, \dots, v_k\} \mid v_i \in \mathbb{R}^N, k \geq -1, v_i \text{ affine linearly independent}\right\}$.

6.2 Abstract simplicial complexes

Definition. An *abstract simplicial complex* in Δ^n is a subset K of the faces $F(\Delta^n)$ such that $e_I \in K$ whenever e_J is in K and $I \subseteq J$.

Remark. Abstract simplicial complexes are downward-closed sets of faces. They have no intrinsic topology. The set of faces $F(\Delta^n)$ of the n-dimensional simplex Δ^n is an abstract simplicial complex.

Definition. If K is an abstract simplicial complex, its *polyhedron* is $|K| = \bigcup_{e_I \in K} e_I \subseteq \Delta^n$.

Remark. Polyhedra are compact and Hausdorff.

Definition. We define $K_r = \{e_I \in K \mid |I| \leq r + 1\}$ to be the set of faces of dimension at most r. This is called the *r-skeleton* of K.

The r-skeleton is an abstract simplicial complex. Note that

$$\{e_\varnothing\} = K_{-1} \subset K_0 \subset \cdots \subset K_n = K$$

We write $\dim K = \max\{\dim e_I \mid e_I \in K\}$.

Definition. The *vertex set* $V(K)$ is the polyhedron $|K_0|$.

Example. $\boldsymbol{\Delta}^n = F(\Delta^n) = \{e_I \mid I \subseteq \{0, \dots, n\}\}$ is a simplicial complex. Its polyhedron is Δ^n, which is homeomorphic to D^n by radial projection.

Example. $\mathbb{S}^{n-1} = \boldsymbol{\Delta}^n_{n-1} = \{e_i \mid I \subsetneq \{0, \dots, n\}\}$ is a simplicial complex. This has polyhedron $\partial\Delta^n$ by definition of the boundary. This is homeomorphic to S^{n-1} by radial projection.

Definition. Let K, L be abstract simplicial complexes in Δ^n and Δ^m respectively. A *simplicial map* $f : K \to L$ is a map such that there is a simplicial map $|f| : \Delta^n \to \Delta^m$ with $f(e_I) = |f|(e_I)$. Equivalently, there is a map $\hat{f} : \{0, \dots, n\} \to \{0, \dots, m\}$ such that $f(e_I) = e_{\hat{f}(I)}$ and $e_I \in K$ implies $e_{\hat{f}(I)} \in L$.

Remark. The identity map is simplicial. The composition of two simplicial maps is simplicial.

Definition. We say a simplicial map $f : K \to L$ is a *simplicial isomorphism* if f is a bijection, or equivalently, $|f|$ is a bijection or $|f|$ is a homeomorphism, treating $|f|$ as a map $|K| \to |L|$.

6.3 Euclidean simplicial complexes

Recall that $\mathcal{S}(\mathbb{R}^n)$ is the set of Euclidean simplices $[v_0, \dots, v_n]$ where the v_i are affine linearly independent.

Definition. $K \subseteq \mathcal{S}(\mathbb{R}^n)$ is a *Euclidean simplicial complex* if
 (i) K is finite;
 (ii) if $\sigma \in K$ and $\tau \in F(\sigma)$, then $\tau \in K$;
 (iii) if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 \in F(\sigma_1) \cap F(\sigma_2)$, so in particular, $\sigma_1 \cap \sigma_2 \in K$.

If so, we write $|K| = \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^n$ with the subspace topology. We write

$$K_r = \{\sigma \in K \mid \dim \sigma \leq r\}$$

for its r-skeleton, which is a Euclidean simplicial complex.

Proposition. Let $|\varphi| : \Delta^n \to \mathbb{R}^n$ be affine linear, and K' be an abstract simplicial complex in Δ^n, such that $|\varphi||_{|K'|}$ is injective. Then $\varphi(K') = \{|\varphi|(e_I) \mid e_I \in K'\}$ is a Euclidean simplicial complex.

Proof. Property (i) is clear since $F(\Delta^n)$ is finite. For property (ii), note that if $\sigma \in \varphi(K')$, there is $e_I \in K'$ such that $\sigma = |\varphi|(e_I)$. If $\tau \in F(\sigma)$, we have $\tau = |\varphi|(e_J)$ for $e_J \subseteq e_I$. Then $e_J \in K'$ since K' is an abstract simplicial complex. So $\tau = |\varphi|(e_J) = \varphi(K')$.

Suppose $\sigma_1 = |\varphi|(e_{I_1})$ and $\sigma_2 = |\varphi|(e_{I_2})$ where $e_{I_1}, e_{I_2} \in K'$. Then $\sigma_1 \cap \sigma_2 = |\varphi|(e_{I_1}) \cap |\varphi|(e_{I_2}) = |\varphi|(e_{I_1} \cap e_{I_2})$ by injectivity. This is equal to $|\varphi|(e_{I_1 \cap I_2}) \in F(\sigma_1) \cap F(\sigma_2)$. □

Definition. We say that the Euclidean simplicial complex $\varphi(K')$ is a *realisation* of an abstract simplicial complex K' in Δ^n, if $|\varphi| : \Delta^n \to \mathbb{R}^n$ is affine linear and injective on $|K'|$.

Remark. If $\varphi(K')$ is a realisation of K', $|\varphi||_{|K'|}$ is injective, so $|\varphi| : |K'| \to |\varphi(K)|$ is a homeomorphism.

Proposition. Let $K \subseteq \mathbb{R}^N$ be a Euclidean simplicial complex. Then $K = \varphi(K')$ for some abstract simplicial complex K', and $|\varphi| : |K'| \to |K|$. Any two K' are related by a simplicial isomorphism.

Informally, every Euclidean simplicial complex is the realisation of some abstract simplicial complex.

Proof. Let $V(K) = |K_0| = \{v_0, \dots, v_n\} \subset \mathbb{R}^N$ be the vertex set of the Euclidean simplicial complex. Define $K' = \left\{e_{\{i_0,\dots,i_k\}} \mid [v_{i_0}, \dots, v_{i_k}] \in K\right\}$. Let $|\varphi| : \Delta^n \to \mathbb{R}^N$ be given by $|\varphi|(e_i) = v_i$.

We show that $|\varphi||_{|K'|}$ is injective. If $\sigma = [v_{i_0}, \dots, v_{i_k}] \in K$, we have that $v_{i_0}, \dots, v_{i_k}$ are affine linearly independent since K is a Euclidean simplicial complex. Then $|\varphi|_{e_I}$ is injective.

Suppose $|\varphi|(p) = |\varphi|(q) = x \in \mathbb{R}^N$, where $p \in e_I \in K'$ and $q \in e_J \in K'$. Then $x \in |\varphi|(e_I) \cap |\varphi|(e_J)$, which is the intersection of simplices in K, so $x \in |\varphi|(e_{I'})$ for $I' \subseteq I \cap J$. Since $|\varphi||_{e_I}$ and $|\varphi||_{e_J}$ are injective, we must have $p, q \in e_{I'}$. But $|\varphi||_{e_{I'}}$ is also injective, so $p = q$. □

Definition. A *simplicial map of Euclidean simplicial complexes* is a map $f : K_1 \to K_2$ if there are realisations $\varphi_i : K_i' \to K_i$ and a simplicial map of abstract simplicial complexes $f' : K_1' \to K_2'$ so that the following diagram commutes.

$$\begin{array}{ccc} K_1' & \xrightarrow{f'} & K_2' \\ {\scriptstyle \varphi_1}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_2} \\ K_1 & \xrightarrow[f]{} & K_2 \end{array}$$

Remark. The composition of simplicial maps of Euclidean simplicial complexes is also a simplicial map.

6.4 Boundaries and cones

Definition. Let σ be an n-dimensional Euclidean simplex. Let $F(\sigma)$ be the set of faces of σ, a Euclidean simplicial complex with $|F(\sigma)| = \sigma$. Let $\not\partial\sigma = F(\sigma)_{n-1} = F(\sigma) \setminus \{\sigma\}$, a Euclidean simplicial complex. Let $\partial\sigma = |\not\partial\sigma| \subset \mathbb{R}^N$ be the boundary of σ. It is homeomorphic to S^{n-1}. Let $\sigma^\circ = \sigma \setminus \partial\sigma$ be the interior of σ.

Definition. Let $X \subseteq \mathbb{R}^N$ and $p \in \mathbb{R}^N$. We say p is *independent* of X if for each $x \in X$, the ray px from p to x has $px \cap X = \{x\}$.

Definition. If p is independent of X, the *cone* is defined by

$$C_p(X) = \{tp + (1-t)x \mid t \in [0,1], x \in X\}$$

Example. Let $X = [v_0, \dots, v_n]$ be an n-simplex. Then p is independent of X if and only if $\{v_0, \dots, v_n, p\}$ is an affine linearly independent set. If so, $C_p(X) = [v_0, \dots, v_n, p]$.

Definition. Let K be a Euclidean simplicial complex in $\mathbb{R}^N$ and p be independent of $|K|$. Then we define the *cone*

$$C_p(K) = K \cup \{[v_0, \dots, v_j, p] \mid [v_0, \dots, v_j] \in K\}$$

Lemma. If p is independent of $|K|$, then $C_p(K)$ is a Euclidean simplicial complex and $|C_p(K)| = C_p(|K|)$.

6.5 Barycentric subdivision

Definition. If $\sigma = [v_0, \dots, v_n]$ is an n-simplex in $\mathbb{R}^N$, we define its *barycentre*

$$b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

Lemma. b_σ is independent of $\partial\sigma$, and $C_{b_\sigma}(\partial\sigma) = \sigma$.

We will define maps β from $\mathcal{S}(\mathbb{R}^N)$ to the set of Euclidean simplicial complexes in $\mathbb{R}^N$, and B from the set of Euclidean simplicial complexes in $\mathbb{R}^N$ to Euclidean simplicial complexes in $\mathbb{R}^N$, satisfying $|\beta(\sigma)| = \sigma$ and $|B(K)| = |K|$. The maps β and B are called *barycentric subdivision*. In order to do this, we will inductively define β and B on simplices and Euclidean simplicial complexes of dimension at most n, and prove the following theorems.

Theorem (first inductive hypothesis)**.** Let $\sigma \in \mathcal{S}(\mathbb{R}^N)$ be an n-simplex. Then $\beta(\sigma)$ is a Euclidean simplicial complex of dimension n, and $|\beta(\sigma)| = \sigma$. If τ is a face of σ and $\sigma_1 \in \beta(\sigma)$ then $\sigma_1 \cap \tau \in \beta(\tau)$.

Theorem (second inductive hypothesis)**.** Let K be an n-dimensional Euclidean simplicial complex. Then $B(K)$ is an n-dimensional Euclidean simplicial complex with polyhedron $|B(K)| = |K|$.

For the base case, let $n = -1$. The only -1-dimensional simplex is $\varnothing$. We define $\beta(\varnothing) = \{\varnothing\}$. The only -1-dimensional simplicial complex is $\{\varnothing\}$, and we define $B(\{\varnothing\}) = \{\varnothing\}$. Both inductive hypotheses hold for this case.

In general, suppose β and B are defined on $n-1$-dimensional simplices and simplicial complexes and that both inductive hypotheses hold. We now define $\beta(\sigma) = C_{b_\sigma}(B(\not\partial\sigma))$ and $B(K) = \bigcup_{\sigma\in K} \beta(\sigma)$.

Example. Let σ be a zero-dimensional simplex. Then $\beta_\sigma(\sigma) = \sigma$.

Example. Let σ be the one-dimensional simplex. $\not\partial\sigma$ is two points p_1, p_2 and the empty set. Then $B(\not\partial\sigma) = \{\varnothing, p_1, p_2\}$. Therefore, $C_p(B(\not\partial\sigma)) = \{\varnothing, p, p_1, p_2, pp_1, pp_2\}$.

Example. Let σ be a two-dimensional simplex with vertices p_1, p_2, p_3. Then $C_p(B(\not\partial\sigma))$ has six 2-simplices, twelve 1-simplices, seven 0-simplices and one empty simplex.

Proof of first inductive hypothesis. $\not\partial\sigma$ is a Euclidean simplicial complex of dimension $n-1$, hence $B(\not\partial\sigma)$ is a Euclidean simplicial complex by the second inductive hypothesis, and $|B(\not\partial\sigma)| = |\not\partial\sigma| = \partial\sigma$. By the lemmas above, b_σ is independent of $\partial\sigma = |B(\not\partial\sigma)|$, so $C_{b_\sigma}(B(\not\partial\sigma))$ is a Euclidean simplicial complex with polyhedron $\left|C_{b_\sigma}(B(\not\partial\sigma))\right| = C_{b_\sigma}(\partial\sigma) = \sigma$. The next part follows from the lemma: if $\sigma \in C_p(K)$, then $\sigma \cap |K| \in K$. □

Proof of second inductive hypothesis. We check the properties required for a Euclidean simplicial complex for $B(K) = \bigcup_{\sigma\in K}\beta(\sigma)$. $\beta(\sigma)$ is finite for each σ and K is finite, so $B(K)$ is finite. If $\sigma \in B(K)$ then $\sigma \in \beta(\sigma')$ for some $\sigma' \in K$, so if $\tau \in F(\sigma)$, then $\tau \in \beta(\sigma')$ since $\beta(\sigma')$ is a Euclidean simplicial complex, so $\tau \in B(K)$, so the second property holds. Suppose $\sigma_1, \sigma_2 \in B(K)$ where $\sigma_i \in \beta(\sigma_i')$ and $\sigma_i' \in K$. Then $\sigma_1 \cap \sigma_2 \subseteq \sigma_1' \cap \sigma_2' = \tau$ since $|\beta(\sigma_i')| = \sigma_i'$, where $\tau \in K$ since K is a Euclidean simplicial complex. Then $\sigma_1 \cap \tau, \sigma_2 \cap \tau \in \beta(\tau)$ by the second part of the first inductive hypothesis. In particular, $\beta(\tau)$ is a Euclidean simplicial complex, so $\sigma_1 \cap \sigma_2 = \underbrace{(\sigma_1 \cap \tau)}_{\in\beta(\tau)} \cap \underbrace{(\sigma_2 \cap \tau)}_{\in\beta(\tau)} \in \beta(\tau) \subseteq B(K)$, so the third property holds. So K is a Euclidean simplicial complex. Now, by the first inductive hypothesis, $|B(K)| = \bigcup_{\sigma\in K}\beta(\sigma) = \bigcup_{\sigma\in K}\sigma = |K|$. □

Lemma. Let $\sigma \in \mathcal{S}(\mathbb{R}^N)$ and $x, v \in \sigma$. Then $\|v - x\| \leq \max_{v_i\in V(\sigma)}\|v - v_i\|$.

Proof. We can write $x = \sum x_i v_i$, where $\sum x_i = 1$, $x_i \geq 0$, and $v_i \in V(\sigma)$. But also, $v = \sum x_i v$. Hence,

$$\|v - x\| = \left\|\sum x_i(v - v_i)\right\| \leq \sum x_i\|v - v_i\| \leq \sum x_i \max\|v - v_i\| = \max\|v - v_i\|$$

□

Applying this twice, $\|x - v\| \leq \max_{v_i\in V(\sigma)}\|v - v_i\| \leq \max_{v_i,v_j\in V(\sigma)}\left\|v_i - v_j\right\|$.

Definition. The *mesh* of a simplex $\sigma \in \mathcal{S}(\mathbb{R}^N)$ is

$$\mu(\sigma) = \max_{v_i,v_j\in V(\sigma)}\left\|v_i - v_j\right\| = \max_{x,v\in\sigma}\|v - x\|$$

If K is a Euclidean simplicial complex, its mesh is $\mu(K) = \max_{\sigma\in K}\mu(\sigma)$.

Lemma. Let b_σ be the barycentre of σ, so $b_\sigma = \frac{1}{n+1}\sum_{i=0}^n v_i$ for $\sigma = [v_0, \dots, v_n]$. Then $\max_{v\in\sigma}\|b_\sigma - v\| \leq \frac{n}{n+1}\mu(\sigma)$.

Proof. $\|b_\sigma - v\| \leq \max_{v_i\in V(\sigma)}\|b_\sigma - v_i\|$. We have

$$\|b_\sigma - v_i\| = \frac{1}{n+1}\left\|\sum_{j\neq i} v_j - nv_i\right\| \leq \frac{1}{n+1}\sum_{j\neq i}\left\|v_j - v_i\right\| \leq \frac{1}{n+1}\cdot n\mu(\sigma)$$

□

Corollary. Let σ be a Euclidean simplex of dimension n. Then $\mu(\beta(\sigma)) \leq \frac{n}{n+1}\mu(\sigma)$. Let K be a Euclidean simplicial complex of dimension n. Then $\mu(B(K)) \leq \frac{n}{n+1}\mu(K)$.

Proof. Let $\tau \in \beta(\sigma)$. Suppose $\tau \in B(\partial\sigma)$. Then, $\mu(\tau) \leq \frac{n-1}{n}\mu(B(\partial\sigma)) \leq \frac{n}{n+1}\mu(\sigma)$ by induction. Otherwise, $\tau = [v_0, \dots, v_k, b_\sigma]$, where $[v_0, \dots, v_k] \in B(\partial\sigma)$. Then $\left\| v_i - v_j \right\| \leq \frac{n-1}{n}\mu(\sigma)$ by induction, and $\| v_i - b_\sigma \| \leq \frac{n}{n+1}\mu(\sigma)$ by the lemma. □

6.6 Simplicial approximation

Lemma. (i) Let $x \in \Delta^n$. Then there exists a unique $I \subseteq \{0, \dots, n\}$ such that $x \in e_I^\circ$.
(ii) If $x \in e_I^\circ$, then $x \in e_J$ if and only if $I \subseteq J$, or equivalently, $e_I \subseteq e_J$.
(iii) Let K be an abstract simplicial complex in Δ^n, and let $x \in e_I^\circ$. Suppose that $x \in |K|$. Then $e_I \in K$.

Proof. *Part (i).* Let $I = \{i \in \{0, \dots, n\} \mid x_i \neq 0\}$. *Part (ii).* Follows from part (i).

Part (iii). $x \in |K|$ implies $x \in e_J$ for some $e_J \in K$. By part (ii), we have $e_I \subseteq e_J$. Since K is an abstract simplicial complex and $e_J \in K$, we have $e_I \in K$. □

Corollary. Let K be a Euclidean simplicial complex, and $x \in |K|$. Then there exists a unique $\sigma \in K$ with $x \in \sigma^\circ$.

Proof. Let $\varphi : K' \to K$ be a realisation of K, so K' is an abstract simplicial complex and φ is a bijection inducing a homeomorphism on the polyhedra. Let $x' = |\varphi^{-1}|(x) \in |K|$. Then x' lies in the interior of a unique e_I by part (i) of the lemma above. Note that $e_I \in K'$ by part (iii), so $\varphi(e_I)$ is the unique $\sigma \in K$ with $x \in \sigma^\circ$. □

Definition. Let K be a Euclidean simplicial complex, and let $v \in V(K)$. Then the *star* $\mathrm{St}_K(v)$ is $\bigcup_{\{\sigma \in K \mid v \in \sigma\}} \sigma^\circ$.

Lemma. (i) Let $x \in |K|$ and $x \in \sigma^\circ$. Then $x \in \mathrm{St}_K(v)$ if and only if $v \in V(\sigma)$.
(ii) $\mathrm{St}_K(v) = |K| \setminus \bigcup_{\{\sigma \in K \mid v \notin V(\sigma)\}} \sigma^\circ = |K| \setminus \bigcup_{\{\sigma \in K \mid v \notin V(\sigma)\}} \sigma$.
(iii) $\{\mathrm{St}_K(v) \mid v \in V(K)\}$ is an open cover of $|K|$.

Proof. *Part (i).* Follows from the fact that if $x \in |K|$, x lies in a unique interior of σ for $\sigma \in K$.

Part (ii). The first equality follows from part (i). The second follows from the fact that if $\tau \in F(\sigma)$ and $v \notin V(\sigma)$, then $v \notin V(\tau)$.

Part (iii). Part (ii) exhibits $\mathrm{St}_K(v)$ as the complement of a finite union of closed sets in $|K|$, so it is open. If $x \in |K|$, then $x \in \sigma^\circ$ for some σ, and if $v \in V(\sigma)$, then $x \in \mathrm{St}_K(v)$, so it is a cover. □

Definition. Let K, L be Euclidean simplicial complexes. Let $f : |K| \to |L|$ be a continuous map, and let $\hat{g} : V(K) \to V(L)$. We say that $\hat{g}$ is a *simplicial approximation* of f if $f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(\hat{g}(v))$ for all $v \in V(K)$.

Theorem. Let $\varphi : K' \to K$ be a realisation of a Euclidean simplicial complex K, and let L be a Euclidean simplicial complex in $\mathbb{R}^M$. We define $g' : |K'| \to \mathbb{R}^M$ to be the affine linear map with $|g'|(v) = \hat{g}(\varphi(v))$ if $v \in V(K')$. Let $|g| = |g'| \circ |\varphi|^{-1}$. Then $|g|$ defines a simplicial map $g : K \to L$, and $|g| \sim f$.

Proof. Let $\sigma \in K$. We must show that $|g|(\sigma) \in L$. Let $x \in \sigma^\circ$ be an arbitrary point in the interior. Then $f(x) \in |L|$, so $f(x) \in \tau^\circ$ with $\tau \in L$. Then $x \in \bigcap_{v \in V(\sigma)} \mathrm{St}_K(v)$, so $f(x) \in \bigcap_{v \in V(\sigma)} f(\mathrm{St}_K(v)) \subseteq \bigcap_{v \in V(\sigma)} \mathrm{St}_L(g(v))$ since g is a simplicial approximation of f. Now, if $v \in V(\sigma)$, $f(x) \in \tau^\circ$ and $f(x) \in \mathrm{St}_L(g(v))$, so $g(v) \in \tau$ by part (i) of the lemma above. Hence, every vertex of $|g|(\sigma)$ is a vertex of τ, so $|g|(\sigma)$ is a face of $\tau \in L$, so $|g|(\sigma) \in L$ as required. So $g : K \to L$ is simplicial.

For the second part, we define $H : |K| \times I \to \mathbb{R}^M$ by $H(x,t) = t|g|(x) + (1-t)f(x)$. This is clearly a homotopy in $\mathbb{R}^M$, but we need to show it is a homotopy in $|L|$. Suppose $x \in \sigma^\circ$ and $f(x) \in \tau^\circ$ as before. Then $x = \sum_{v_i \in V(\sigma)} x_i v_i$, so $|g|(x) = \sum_{v_i \in V(\sigma)} x_i |g|(v_i) \in \tau$ since $|g|(v_i) \in \tau$. Since τ is convex, and $|g|(x), f(x) \in \tau$, we must have $H(x,t) \in \tau$ for $t \in [0,1]$. So $H : |K| \times I \to |L|$, which is the desired homotopy. □

Theorem (simplicial approximation theorem)**.** Let K, L be Euclidean simplicial complexes. Let $f : |K| \to |L|$ be a continuous map. Then there exists $r > 0$ and a simplicial map $g : B^r(K) \to L$ such that $|g| \sim f$.

Note that $|B^r(K)| = |K|$, so $|g| : |B^r(K)| \to |L|$ can be thought of as a map $|K| \to |L|$.

Proof. We have the open cover $\{\mathrm{St}_L(v) \mid v \in V(L)\}$ of $|L|$. $f : |K| \to |L|$ is continuous, so $\left\{f^{-1}(\mathrm{St}_L(v)) \mid v \in V(L)\right\}$ is an open cover of $|K|$. Now, $|K|$ is a compact metric space, so we can apply the Lebesgue covering lemma to find $\delta > 0$ and a function $|K| \to V(L)$ mapping x to some vertex v_x such that $B_\delta(x) \subseteq f^{-1}(\mathrm{St}_L(v_x))$. Let r be a natural number such that $\mu(B^r(K)) < \delta$, and let $K' = B^r(K)$. If $\sigma \in K'$ and $x \in V(\sigma)$, then $\sigma \subseteq B_\delta(x)$, since $\mu(K') < \delta$. If $x \in V(K')$, then

$$\mathrm{St}_{K'}(x) = \bigcup_{\{\sigma \mid x \in V(\sigma)\}} \sigma^\circ \subseteq \bigcup_{\{\sigma \mid x \in V(\sigma)\}} \sigma \subseteq B_\delta(x)$$

Hence, $f(\mathrm{St}_{K'}(x)) \subseteq f(B_\delta(x)) \subseteq \mathrm{St}_L(v_x)$, so the function $\hat{g} : V(K') \to V(L)$ given by $\hat{g}(x) = v_x$ is a simplicial approximation of f. So by the previous theorem, $\hat{g}$ determines a simplicial map $g : K' \to L$ with $|g| \sim f$. □

Corollary. Let K, L be Euclidean simplicial complexes, where $\dim K < \dim L$. Let $f : |K| \to |L|$ be continuous. Then $f \sim |g|$ where $|g|$ is not surjective.

Proof. Let $g : B^r(K) \to L$ be a simplicial map such that $f \sim |g|$. Let $k = \dim B^r(K) = \dim K$. Then $|g| : |K| \to |L_k| \subsetneq |L|$ since $\dim L > k$. So $|g|$ is not surjective. □

Remark. It is a general fact that simplicial functions map an i-skeleton into an i-skeleton for each i.

Theorem. If $k < n$, any continuous map $S^k \to S^n$ is null-homotopic.

Proof. $S^k \simeq |\mathbb{S}^k|$ and $S^n \simeq |\mathbb{S}^n|$. By the above corollary, $f \sim |g|$ where $|g| : S^k \to S^n$ is not surjective. Let $|g| : S^k \to S^n \setminus \{p\}$.

$$\begin{array}{ccc} S^k & \xrightarrow{g'} & S^n \setminus \{p\} \\ & \underset{|g|}{\searrow} & \downarrow \iota \\ & & S^n \end{array}$$

But $S^n \setminus \{p\} \simeq \mathbb{R}^n$ is contractible. So g' is null-homotopic, so $|g| \sim \iota \circ g'$ is null-homotopic. □

7 Simplicial homology

7.1 Chain complexes

Definition. A *(finitely generated) chain complex* $(C_\bullet, d)$ is
(i) a collection of free (finitely generated) abelian groups C_i for $i \in \mathbb{Z}$ (and if finitely generated, $C_i = 0$ for all but finitely many i);
(ii) a collection of homomorphisms $d_i : C_i \to C_{i-1}$;
(iii) $d_{i-1} \circ d_i = 0$ for all i.

$$\cdots \xleftarrow{d_{-2}} C_{-2} \xleftarrow{d_{-1}} C_{-1} \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} \cdots$$

Usually, we write $C_\bullet = \bigoplus_i C_i$, and $d = \bigoplus_i d_i : C_\bullet \to C_\bullet$. We can check that $d_{i-1} \circ d_i = 0$ for all i is equivalent to the statement that $d \circ d = d^2 = 0$.

Remark. Free finitely generated abelian groups are isomorphic to $\mathbb{Z}^n$ for some n. A chain complex defined over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{F}_p$ is similar, except that C_i is a vector space over the $\mathbb{Q}, \mathbb{R}, \mathbb{F}_p$ and the d_i are linear maps. Every chain complex determines another chain complex over $\mathbb{Q}, \mathbb{R}, \mathbb{F}_p$ by replacing $\mathbb{Z}^{n_i}$ with $\mathbb{Q}^{n_i}$, for example, and the d_i are given by the same matrices.

Remark. There is a unique group homomorphism to and from the trivial abelian group 0. Arrows to and from this group can therefore be unlabelled.

Example (reduced chain complex of the simplex)**.** Consider the reduced chain complex of Δ^n. We define $\widetilde{C}_k(\Delta^n) = \langle e_I \mid |I| = k+1, I \subseteq \{0, \dots, n\} \rangle$, the free abelian group on a basis given by the e_I. We also define $d(e_I) = \sum_{j=0}^{|I|} (-1)^j e_{I_j}$ where if $I = i_0 i_1 \dots i_k$ and $i_0 < \dots < i_k$, we define $I_j = I \setminus \{i_j\}$. For example, consider $\widetilde{C}_\bullet(\Delta^2)$.

$$\widetilde{C}_2(\Delta^2) = \langle e_{012} \rangle; \quad \widetilde{C}_1(\Delta^2) = \langle e_{01}, e_{02}, e_{12} \rangle; \quad \widetilde{C}_0(\Delta^2) = \{e_0, e_1, e_2\}; \quad \widetilde{C}_{-1}(\Delta^2) = \{e_\varnothing\}$$

and, for example,

$$d(e_{012}) = (-1)^0 e_{12} + (-1)^1 e_{02} + (-1)^2 e_{01} = e_{12} - e_{02} + e_{01}$$

$$d(e_{01}) = e_1 - e_0; \quad d(e_{02}) = e_2 - e_0; \quad d(e_{12}) = e_2 - e_1; \quad d(e_0) = d(e_1) = d(e_2) = e_\varnothing$$

Note that $\widetilde{C}_i(\Delta^2) = 0$ if $i < -1$ or $i > 2$. We have $d^2(e_{012}) = d(e_{12} - e_{02} + e_{01}) = e_2 - e_1 - e_2 + e_0 + e_1 - e_0 = 0$, as required.

$$0 \longleftarrow \widetilde{C}_{-1} \xleftarrow{d_0} \widetilde{C}_0 \xleftarrow{d_1} \widetilde{C}_1 \xleftarrow{d_2} \widetilde{C}_2 \longleftarrow 0$$

Proposition. For $\widetilde{C}_\bullet(\Delta^n)$, $d^2 = 0$.

Proof. The e_I are a basis for $\widetilde{C}_\bullet(\Delta^n)$, so it suffices to check that $d^2(e_I) = 0$ for each I. For some $c_{jj'}$, we have $d^2(e_I) = \sum_{j<j'} c_{jj'} e_{I_{\hat{j},\hat{j}'}}$ where $I_{\hat{j},\hat{j}'} = I \setminus \{i_j, i_{j'}\}$. We can compute that $c_{jj'}$ has a contribution of $(-1)^j(-1)^{j'-1}$ by first considering j then j', since $i_{j'}$ is the $(j'-1)$th element of $I_{\hat{j}}$. Note also that by computing the term in the sum with j, j' in the other order, we have a contribution of $(-1)^{j'}(-1)^i$. Hence $c_{jj'} = (-1)^j(-1)^{j'-1} + (-1)^{j'}(-1)^i = 0$. □

Example (chain complex of the simplex). The chain complex of Δ^n is defined by $C_i(\Delta^n) = \widetilde{C}_i(\Delta^n)$ if $i \geq 0$, but $C_{-1}(\Delta^n) = 0$. This removes the empty face $e_\varnothing$. The d_i are unchanged.

$$0 \longleftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \longleftarrow 0$$

Definition. Let K be an abstract simplicial complex in Δ^n. Let

$$\widetilde{C}_k(K) = \langle e_I \mid |I| = k+1, e_I \in K \rangle \leq \widetilde{C}_k(\Delta^n)$$

Since $e_I \in K$ implies $e_{I_j} \in K$, $d_k : \widetilde{C}_k(K) \to \widetilde{C}_{k-1}(K)$. So $(\widetilde{C}_\bullet(K), d)$ is a chain complex.

Definition. Let $(C_\bullet, d)$ be a chain complex, and let $x \in C_k$. We say that x is a *cycle* or *closed* if $dx = 0$, so $x \in \ker d_k$. We say that x is a *boundary* or *exact* if $x = dy$ for some y, so $x \in \operatorname{Im} d_{k+1}$.

Remark. The statement $d^2 = 0$ is equivalent to the statement $\operatorname{Im} d_{k+1} \subseteq \ker d_k$ for each k, so boundaries are always cycles.

7.2 Homology groups

Definition. Let $(C_\bullet, d)$ be a chain complex. Its *kth homology group* is

$$H_k(C) = {\ker d_k}/{\operatorname{Im} d_{k+1}}$$

Remark. Homology groups are abelian.

Example. Consider $\widetilde{C}_\bullet(\Delta^2)$. Recall $\widetilde{C}_2 = \langle e_{012}\rangle$ and $d(e_{012}) = e_{12} - e_{02} + e_{01}$. Hence $\ker d_2 = 0$ and $\operatorname{Im} d_3 = 0$, so $H_2(\widetilde{C}_\bullet(\Delta^2)) = 0$.

We have $\widetilde{C}_1 = \langle e_{12}, e_{02}, e_{01}\rangle$, and $d(ae_{01} + be_{12} + ce_{02}) = a(e_1 - e_0) + b(e_2 - e_1) + c(e_2 - e_0) = -(a+c)e_0 + (a-b)e_1 + (b+c)e_2$. Hence $ae_{01} + be_{12} + ce_{02} \in \ker d$ if and only if $a = b = -c$. So $x \in \langle e_{12} - e_{02} + e_{01}\rangle = \operatorname{Im} d_2$, giving $H_1(\widetilde{C}_\bullet(\Delta^2)) = 0$.

We have $\widetilde{C}_0 = \langle e_0, e_1, e_2\rangle$ and $d(e_i) = e_\varnothing$, so $\ker d_0 = \{ae_0 + be_1 + ce_2 \mid a + b + c = 0\}$. Conversely, $\operatorname{Im} d_1 = \operatorname{span}\{e_1 - e_0, e_2 - e_0, e_2 - e_1\} = \ker d_0$. So in fact $H_0(\widetilde{C}_\bullet(\Delta^2)) = 0$.

Now $\widetilde{C}_{-1} = \langle e_\varnothing\rangle = \ker d_{-1} = \langle e_\varnothing\rangle = \operatorname{Im} d_0$ so $H_{-1}(\widetilde{C}_\bullet(\Delta^2)) = 0$. So all of the homology groups of $\widetilde{C}_\bullet(\Delta^2)$ are trivial. Note that

$$H_i(C_\bullet(\Delta^2)) = \begin{cases} H_i(\widetilde{C}_\bullet(\Delta^2)) & i > 0 \\ \langle e_0, e_1, e_2\rangle \big/ \operatorname{span}\{e_1 - e_0, e_2 - e_0, e_2 - e_1\} \simeq \mathbb{Z} & i = 0 \end{cases}$$

Definition. Let K be an abstract simplicial complex in Δ^n. Then we define the ith reduced homology group of K to be $\widetilde{H}_i(K) = H_i(\widetilde{C}_\bullet(K))$. Then $C_\bullet(K) = \widetilde{C}_\bullet(K) \big/ \langle e_\varnothing\rangle$ is a chain complex, and $H_i(K) = H_i(C_\bullet(K))$ is the ith homology group of K.

Example. Let $K = \{e_0, e_1, \dots, e_r, e_\varnothing\}$, so $|K|$ is a collection of $r+1$ disjoint points. In this case, $\widetilde{C}_i(K) = 0$ for $i > 0$. $\widetilde{C}_0(K) = \langle e_0, \dots, e_r\rangle$ and $d(e_i) = \varnothing$. $\widetilde{C}_{-1}(K) = \langle e_\varnothing\rangle$. Hence $\ker d_0 = \langle e_1 - e_0, \dots, e_r - e_0\rangle$ and $\operatorname{Im} d_1 = 0$, so $H_0(\widetilde{C}_\bullet(K)) = \mathbb{Z}^r$, and $H_{-1}(\widetilde{C}_\bullet(K)) = 0$. Note that $H_0(C_\bullet(K)) = \mathbb{Z}^{r+1} = \langle e_0, \dots, e_r\rangle$.

Example. Recall that any Euclidean simplicial complex is realised by an abstract simplicial complex, but we have choice in the labelling of the vertices. Let T_n be the boundary of a convex n-gon in $\mathbb{R}^2$. Then the abstract simplicial complex

$$K' = \{e_\varnothing, e_0, \dots, e_{n-1}, e_{01}, e_{12}, \dots, e_{(n-2)(n-1)}, e_{(n-1)0}\}$$

in Δ^{n-1} realises T_n. Then

$$\begin{aligned} C_1(K') &= \langle e_{01}, e_{12}, \dots, e_{(n-2)(n-1)}, e_{(n-1)0}\rangle \\ C_0(K') &= \langle e_0, \dots, e_{n-1}\rangle \end{aligned}$$

We have $d(e_{i(i+1)}) = e_{i+1} - e_i$, so $\ker d_1 = \langle x\rangle$ where

$$x = e_{01} + e_{12} + \dots + e_{(n-2)(n-1)} - e_{0(n-1)}$$

Note that $\operatorname{Im} d_1 = \operatorname{span}\{e_{i+1} - e_i\}$. Hence

$$\begin{aligned} H_1(K') &= \ker d_1 \big/ \operatorname{Im} d_2 = \langle x\rangle \big/ 0 \simeq \mathbb{Z} \\ H_0(K') &= \ker d_0 \big/ \operatorname{Im} d_1 = \langle e_0, \dots, e_{n-1}\rangle \big/ \operatorname{span}\{e_1 - e_0, \dots, e_{n-1} - e_{n-2}\} \simeq \mathbb{Z} \end{aligned}$$

Note that this result does not depend on the choice of n, and $|T_n| \simeq S^1$ also does not depend on n. In fact, $H_\bullet(K)$ depends only on $|K|$.

7.3 Chain maps

Definition. Let $(C_\bullet, d)$ and $(C'_\bullet, d')$ be chain complexes. A *chain map* $f : C_\bullet \to C'_\bullet$ is
(i) for each i, a homomorphism $f_i : C_i \to C'_i$, such that
(ii) $f_{i-1} \circ d_i = d'_i \circ f_i$.

Remark. We can interpret f as $\bigoplus_i f_i : C_\bullet \to C'_\bullet$, given by a block matrix

$$\begin{pmatrix} f_n & & \\ & f_{n-1} & \\ & & \ddots \end{pmatrix}$$

Then part (ii) of the definition is equivalent to the statement $d'f = fd$.

If $x \in \ker d$, we write $[x] \in H_\bullet(C)$ for its image under the map $\ker d \to {}^{\ker d}\!/\!_{\operatorname{Im} d}$.

Remark. $f(\ker d) \subseteq \ker d'$ because if $dx = 0$, we have $d'(f(x)) = f(d(x)) = f(0) = 0$. $f(\operatorname{Im} d) \subseteq \operatorname{Im} d'$, because if $x = dy$, we have $f(x) = f(d(y)) = d'(f(y))$. So f descends to a well-defined homomorphism $f_\star : {}^{\ker d}\!/\!_{\operatorname{Im} d} \to {}^{\ker d'}\!/\!_{\operatorname{Im} d'}$ such that $f_\star([x]) = [f(x)]$. So $f_\star : H_\bullet(C) \to H_\bullet(C')$. This is called the map *induced by* f.

Remark. The composition of two chain maps is a chain map, and $(f \circ g)_\star = f_\star \circ g_\star$.

Let K be an abstract simplicial complex in Δ^n, and L be an abstract simplicial complex in Δ^m. Let $f : K \to L$ be a simplicial map, so it is determined by $\hat{f} : \{0, \dots, n\} \to \{0, \dots, m\}$. We wish to define a chain map $f_\sharp : C_\bullet(K) \to C_\bullet(L)$, which will induce $f_\star : H_\bullet(K) \to H_\bullet(L)$. Perhaps the most obvious guess would be to define $f_\sharp(e_I) = f(e_I) = e_{\hat{f}(I)}$. This is not the correct definition.

First, consider $f : \mathbf{\Delta}^1 \to \mathbf{\Delta}^1$ given by $e_0 \mapsto e_0$, $e_1 \mapsto e_0$. Then $f(e_{01}) = e_0$, but $e_{01} \in C_1(\mathbf{\Delta}^1)$ and $e_0 \in C_0(\mathbf{\Delta}^1)$. So f does not preserve grading, and hence cannot be a chain map.

Consider also $f : \mathbf{\Delta}^1 \to \mathbf{\Delta}^1$ given by $e_0 \mapsto e_1$ and $e_1 \mapsto e_0$. Now, $f(e_{01}) = e_{01}, f(e_0) = e_1, f(e_1) = e_0$, so $df(e_{01}) = d(e_{01}) = e_1 - e_0$ but $fd(e_{01}) = f(e_1 - e_0) = e_0 - e_1$.

The solution to both problems is to change our perspective on the indices I. Until now, we have defined $I \subseteq \{0, \dots, n\}$ and written $I = i_0 i_1 \dots i_k$ where $i_0 < \dots < i_k$. Instead, we will allow $I \in \{0, \dots, n\}^{k+1}$, so $I = (i_0, i_1, \dots, i_k) = i_0 i_1 \dots i_k$ with no restriction on order. For instance, e_{00}, e_{10} are permitted.

We impose relations on the set of all such I to form an abelian group generated by equivalence classes of the $\{0, \dots, n\}^{k+1}$. We will define that $e_I = -e_{I'}$ when I, I' are related by switching two indices; so $e_{102} = -e_{012} = e_{210}$. If e_I contains a repetition, we require $e_I = 0$.

More concretely, if $I \in \{0, \dots, n\}^{k+1}$, let I' be the unique ordered permutation of I if I has no repetitions. Then $e_I = (-1)^{S(I)} e_{I'}$ if I has no repetitions, and $e_I = 0$ if I has a repetition, where $(-1)^{S(I)}$ is the sign of the permutation $\sigma \in S^{k+1}$ mapping I to I'. If we draw I and I' in order as a bipartite planar graph, connected by matching labels, $S(I)$ is the number of crossings.

Lemma. Let $i_j \in I$, and suppose i_j is in position $i_{j'}$ in I'. Then $S(I) - S(I_j) \equiv j - j' \bmod 2$.

Proposition. Let $I \in \{0, \dots, n\}^{k+1}$. Then $d(e_I) = \sum_{j=0}^{k} (-1)^j e_{I_j}$, where I_j is obtained from I by omitting the jth entry in the tuple I.

We have already defined d for ordered sequences of indices; this proposition states that this formula holds for all sequences of indices.

Proof.

$$\sum_{j=0}^{k}(-1)^k e_{I_j} = \sum_{j=0}^{k}(-1)^j(-1)^{S(I_j)}e_{I'_j} = \sum_{j=0}^{k}(-1)^{j'}(-1)^{S(I)}e_{(I')_j} = (-1)^{S(I)}d(e_{I'}) = d(e_I)$$

□

Example. $d(e_{210}) = (-1)^0 e_{10} + (-1)^1 e_{20} + (-1)^2 e_{21} = -e_{01} + e_{02} - e_{12} = d(-e_{012})$, where by definition, $e_{210} = -e_{012}$ so $d(e_{210}) = -d(e_{012})$.

Definition. Let $f : K \to L$ be a simplicial map. Then $f_\sharp : C_k(K) \to C_k(L)$ is defined by $f_\sharp(e_I) = e_{\hat{f}(I)}$ where if $I = (i_0, \dots, i_k)$ we define $\hat{f}(I) = (\hat{f}(i_0), \dots, \hat{f}(i_k))$.

This definition of $f_\sharp$ preserves grading.

Proposition. $f_\sharp$ is a chain map.

Proof.

$$d(f_\sharp(e_I)) = d(e_{\hat{f}(I)}) = \sum_{j=0}^{k}(-1)^j e_{(\hat{f}(I))_j} = f_\sharp\left(\sum_{j=0}^{k}(-1)^j e_{I_j}\right) = f_\sharp(d(e_I))$$

□

Example. Let $f : \mathbf{\Delta}^1 \to \mathbf{\Delta}^1$ be the simplicial map defined by $f(e_0) = e_0$ and $f(e_1) = e_0$. Then $f_\sharp(e_{01}) = e_{00} = 0$.

Now let $f(e_0) = e_1$ and $f(e_1) = e_0$. Then $f_\sharp(e_{01}) = e_{10} = -e_{01}, f_\sharp(e_0) = e_1, f_\sharp(e_1) = e_0$. So $d(f_\sharp(e_{01})) = -d(e_{01}) = e_0 - e_1 = f(d(e_{01}))$.

7.4 Chain homotopies

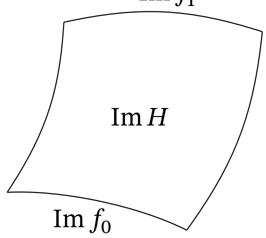
Definition. Let $f_0, f_1 : (C, d) \to (C', d')$ be chain maps. Then f_0 is *chain homotopic* to f_1, written $f_0 \sim f_1$, if there are

(i) homomorphisms $h_i : C_i \to C'_{i+1}$, where we write $h = \bigoplus_i h_i$, satisfying

(ii) $d'h + hd = f_0 - f_1$.

In this case, we say h is the *chain homotopy*.

Example. Suppose $f_0, f_1 : X \to Y$ are homotopic maps via H. Suppose $X = \Delta^n$.



Here,

$$\partial(H(\Delta^n)) = H(\partial\Delta^n) \cup f_1(\Delta^n) \cup f_0(\Delta^n) \implies \partial H + H\partial = f_1 + f_0$$

without considering signs.

Lemma. If $f_1 \sim f_0$, then $f_{1\star} = f_{0\star} : H_\bullet(C) \to H_\bullet(C')$.

Proof. Let $[x] \in H_\bullet(C)$. Then $dx = 0$. So

$$f_{1\star}[x] - f_{0\star}[x] = [(f_1 - f_0)x] = [(d'h + hd)x] = [d'h(x)] = 0$$

since $d'h(x) \in \operatorname{Im} d'$. □

Definition. We say a chain complex (C, d) is *contractible* if $\mathrm{id}_C \sim 0_C$, where 0_C is the zero map.

Lemma. Let (C, d) be contractible. Then $H_\bullet(C) = 0$.

Proof. Let $[x] \in H_\bullet(C)$. Then $[x] = \mathrm{id}_\star[x] = 0_\star[x] = [0]$. So $H_k(C)$ is the trivial group for each k. □

Definition. Let K be an abstract simplicial complex in Δ^n. Let $e_0 \notin K$. Then the *cone* is $C_{e_0}(K) = K \cup \{e_{0I} \mid e_I \in K\}$.

Remark. $C_{e_0}(K)$ is an abstract simplicial complex. If K' is a realisation of K, where $e_0 \notin K$ and K' is independent of p, then $C_p(K')$ is a realisation of $C_{e_0}(K)$.

Example. Consider $\hat{\mathbf{\Delta}}^n = \{e_I \in \mathbf{\Delta}^{n+1} \mid 0 \notin I\} \simeq \mathbf{\Delta}^n$. Then $C_{e_0}(\hat{\mathbf{\Delta}}^n) = \mathbf{\Delta}^{n+1}$.

Proposition. $\widetilde{C}_\bullet(C_{e_0}(K))$ is contractible.

Proof. Define $h : \widetilde{C}_k(C_{e_0}(K)) \to \widetilde{C}_{k+1}(C_{e_0}(K))$ by $h(e_I) = e_{0I}$. Note that if $0 \in I$, then $e_{0I} = 0$.

If $0 \in I$ then $dh(e_I) = 0$, and $hd(e_I) = h\left(\sum_{j=0}^k (-1)^j e_{I_j}\right) = h(e_{I \setminus \{0\}} + \sum e_{I'})$ where $0 \in I'$. Then $hd(e_I) = e_I + 0 = e_I$. Otherwise, if $0 \notin I$, then $dh(e_I) = d(e_{0I}) = e_I + \sum_{j=0}^k (-1)^{k+1} e_{0I_j} = e_I - h(de_I)$. In either case, $dh + hd = \mathrm{id}$. □

Corollary. $\widetilde{H}_\bullet(C_{e_0}(K)) = 0$. In particular,

$$H_i(C_{e_0}(K)) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Proof. Let $\widetilde{C}_\bullet(C_{e_0}(K)) = (\widetilde{C}, \widetilde{d})$, and $C_\bullet(C_{e_0}(K)) = (C, d)$. The first part follows from the previous result. For the second part, note that $\widetilde{H}_{-1}(C_{e_0}(K)) = 0$, so $\widetilde{d}_0 : \widetilde{C}_0 \to \widetilde{C}_{-1} = \langle e_\varnothing \rangle \simeq \mathbb{Z}$ is surjective. So $\mathbb{Z} \simeq \operatorname{Im} \widetilde{d}_0 \simeq {}^{\widetilde{C}_0}\!/\!_{\ker \widetilde{d}_0} \simeq {}^{\widetilde{C}_0}\!/\!_{\operatorname{Im} \widetilde{d}_1}$ since $\widetilde{H}_0(C) = 0$. But ${}^{\widetilde{C}_0}\!/\!_{\operatorname{Im} \widetilde{d}_1} \simeq {}^{C_0}\!/\!_{\operatorname{Im} d_1} = {}^{\ker d_0}\!/\!_{\operatorname{Im} d_1} = H_0(C_{e_0}(K))$. For $i \geq 0$, note that $\ker \widetilde{d}_i = \ker d_i$ and $\operatorname{Im} \widetilde{d}_{i+1} = \operatorname{Im} d_{i+1}$. Hence $H_i(\widetilde{C}) = H_i(C)$ for $i > 0$. □

Definition. Let $\mathbb{S}^n = \mathbf{\Delta}^n \setminus e_{0\dots n}$. Then

$$H_i(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Similar to the previous proof, but now we remove the 'top' generator instead of the 'bottom' one. □

Alternatively, we can prove this fact using the results from the next subsection.

7.5 Exact sequences

Definition. Let A_k be a sequence of abelian groups for $k \in \mathbb{Z}$, and $f_k : A_k \to A_{k-1}$ be homomorphisms. We say that the sequence is *exact* at A_k if $\ker f_k = \operatorname{Im} f_{k+1}$. If it is exact at all A_k, we say the sequence is exact.

$$\cdots \xrightarrow{f_{k+2}} A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} \cdots$$

Example.

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact at A if and only if f is injective.

$$B \xrightarrow{g} C \longrightarrow 0$$

is exact at C if and only if g is surjective.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if f is injective, g is surjective, and $g : {}^{B}\!/\!_{\operatorname{Im} f} \to C$ is an isomorphism, so $C \simeq {}^{B}\!/\!_{\operatorname{Im} f}$. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence*.

Definition. Let $g : B \to C$. Then the *cokernel* of g is $\operatorname{coker} g = {}^{C}\!/\!_{\operatorname{Im} g}$.

In general, a sequence is exact if and only if

$$0 \longrightarrow \operatorname{coker} f_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} \ker f_{k-1} \longrightarrow 0$$

is a short exact sequence for every k.

Definition. A short exact sequence of chain complexes is a short exact sequence

$$0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$$

where $A_\bullet, B_\bullet, C_\bullet$ are chain complexes, and f, g are chain maps.

Equivalently, we have the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\
0 & \longrightarrow & A_k & \xrightarrow{f} & B_k & \xrightarrow{g} & C_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{f} & B_{k-1} & \xrightarrow{g} & C_{k-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\
0 & \longrightarrow & A_{k-2} & \xrightarrow{f} & B_{k-2} & \xrightarrow{g} & C_{k-2} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_A} & & \downarrow{\scriptstyle d_B} & & \downarrow{\scriptstyle d_C} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where all squares commute since f, g are chain maps, and all rows are exact.

Lemma (snake lemma)**.** Let $0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$ be a short exact sequence of chain complexes. Then there is an exact sequence

$$H_k(A) \xrightarrow{f_\star} H_k(B) \xrightarrow{g_\star} H_k(C) \xrightarrow{\partial_k} H_{k-1}(A) \xrightarrow{f_\star} H_{k-1}(B) \xrightarrow{g_\star} H_{k-1}(C)$$

The homomorphism ∂_k is called the *connecting homomorphism*. Since this exists for all k, this gives a long exact sequence of homology groups.

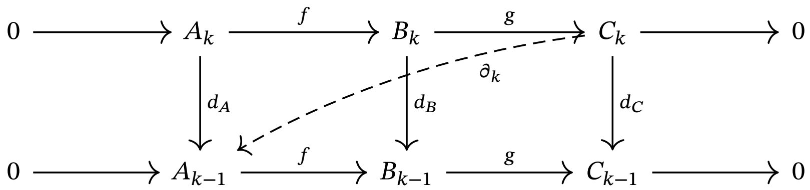
Proof. Let $[c] \in H_k(C)$, so $dc = 0$. Then,

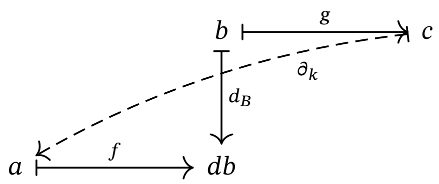
(i) g is surjective, so we can choose $b \in B_k$ such that $g(b) \in c$.

(ii) $g(db) = dg(b) = dc = 0$, so $db \in \ker g$. Since the sequence is exact at B, we have $db = f(a)$ for some $a \in A_{k-1}$.

(iii) $f(da) = d(fa) = d^2(b) = 0$. Since f is injective, $da = 0$.

We then define $\partial_k[c] = [a] \in H_{k-1}(A)$. To visualise the above argument, the following diagrams can be overlaid; the first diagram shows the groups, and the second diagram shows the corresponding elements.





This definition does not depend on any choices that we made; for example, $[c] = [c']$ implies $\partial_k[c] = \partial_k[c']$.

(i) If $g(b') = c$, then $g(b - b') = 0$. By exactness, $b - b' = f(\alpha)$. Then $db - db' = f(d(\alpha))$. Let $f(a) = db$ and $f(a') = db'$. So $a - a' = d\alpha$, so $[a] = [a']$.

(ii) Suppose $[c] = [c']$. Then $c - c' = d\gamma$ for $\gamma \in C_{k+1}$. g is surjective, so let $\gamma = g(\beta)$. Then $b - b' = d\beta$, so $db = db'$. Since $a = a'$, we have $[a] = [a']$.

We need to check exactness. We will show that ker $\subseteq$ Im in each case, the other direction is left as an exercise.

(i) Consider $H_k(C)$. If $\partial_k[c] = 0$, then $a = d\alpha$ for $\alpha \in A_k$. Then $d(f(\alpha)) = f(d\alpha) = f(a) = db$. So $d(b - f(\alpha)) = 0$, giving $[b - f(\alpha)] \in H_k(B)$. Then $g_\star[b - f(\alpha)] = [g(b) - g(f(\alpha))] = [g(b)] = [c]$ by exactness. So $[c] \in \operatorname{Im} g_\star$ as required.

(ii) Consider $H_k(B)$. If $g_\star[b] = 0$, then $g(b) = d\gamma$ for some $\gamma \in C_{k+1}$. g is surjective, so $\gamma = g(\beta)$ for $\beta \in B_{k+1}$. Then $g(b - d\beta) = c - dg(\beta) = c - c = 0$, so $b - d\beta = f(\alpha)$ for $\alpha \in A_k$. So $f(d\alpha) = df(\alpha) = db - d^2\beta = 0$. Hence $[b] = [b - d\beta] = f_\star[\alpha]$. So $[\beta] \in \operatorname{Im} f_\star$.

(iii) Consider $H_{k-1}(A)$. If $f_\star[a] = 0$, then $f(a) = db$ for some b in B_{k-1}. Then $[a] = \partial_k[g(b)]$ since $dg(b) = g(db) = g(f(a)) = 0$. So $[a] \in \operatorname{Im} \partial_k$.

□

Example. Let $B = C_\bullet(\boldsymbol{\Delta}^n)$, and $A = C_\bullet(\mathbb{S}^{n-1})$. Let C be defined by

$$C_k = \begin{cases} \langle e_{0\dots n} \rangle & k = n \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$H_k(C) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}$$

Let $n > 1$. Then we have a short exact sequence $0 \longrightarrow \mathbb{S}^{n-1} \xrightarrow{f} \mathbf{\Delta}^n \xrightarrow{g} C \longrightarrow 0$ and hence we have

$$H_k(\mathbb{S}^{n-1}) \xrightarrow{f_\star} H_k(\mathbf{\Delta}^n) \xrightarrow{g_\star} H_k(C) \xrightarrow{\partial_k} H_{k-1}(\mathbb{S}^{n-1}) \xrightarrow{f_\star} H_{k-1}(\mathbf{\Delta}^n) \xrightarrow{g_\star} H_{k-1}(C)$$

Now, letting $k = n$, we can therefore find the exact sequence

$$H_n(\mathbb{S}^{n-1}) \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_k} H_{n-1}(\mathbb{S}^{n-1}) \longrightarrow 0 \longrightarrow 0$$

By exactness at $\mathbb{Z}$ and $H_{n-1}(\mathbb{S}^{n-1})$, ∂_k is an isomorphism. Hence $H_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$.

7.6 Mayer–Vietoris sequence

Let K_1, K_2 be abstract simplicial complexes in Δ^n. Then $K_1 \cap K_2$ and $K_1 \cup K_2$ are also abstract simplicial complexes in Δ^n. We have the following commutative square of simplicial maps given by inclusion.

$$\begin{array}{ccccc} & & K_1 & & \\ & \overset{i_1}{\nearrow} & & \overset{j_1}{\searrow} & \\ K_1 \cap K_2 & & & & K_1 \cup K_2 \\ & \underset{i_2}{\searrow} & & \underset{j_2}{\nearrow} & \\ & & K_2 & & \end{array}$$

This induces a commutative square of chain maps as shown.

$$\begin{array}{ccccc} & & C_\bullet(K_1) & & \\ & \overset{i_{1\sharp}}{\nearrow} & & \overset{j_{1\sharp}}{\searrow} & \\ C_\bullet(K_1 \cap K_2) & & & & C_\bullet(K_1 \cup K_2) \\ & \underset{i_{2\sharp}}{\searrow} & & \underset{j_{2\sharp}}{\nearrow} & \\ & & C_\bullet(K_2) & & \end{array}$$

Proposition. Let K_1, K_2 be abstract simplicial complexes in Δ^n. Then the sequence

$$0 \longrightarrow C_\bullet(K_1 \cap K_2) \xrightarrow{i} C_\bullet(K_1) \oplus C_\bullet(K_2) \xrightarrow{j} C_\bullet(K_1 \cup K_2) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$i = \begin{pmatrix} i_{1\sharp} \\ i_{2\sharp} \end{pmatrix}; \quad j = \begin{pmatrix} j_{1\sharp} & -j_{2\sharp} \end{pmatrix}$$

Proof. We must check exactness at each location. $i_{1\sharp}$ is injective, so i is injective.

If $j((a,b)) = 0$, then $j_{1\sharp}(a) = j_{2\sharp}(b)$, so $a = b \in C_\bullet(K_1) \cap C_\bullet(K_2) = C_\bullet(K_1 \cap K_2)$. Hence $(a,b) = i(a)$, so $\ker j \subseteq \operatorname{Im} i$. For the other direction, $gf(a) = (j_{1\sharp} \circ i_{1\sharp})(a) - (j_{2\sharp} \circ i_{2\sharp}(a)) = 0$ since the square of inclusion maps commutes. So $\operatorname{Im} i \subseteq \ker j$, so the sequence is exact at $C_\bullet(K_1) \oplus C_\bullet(K_2)$.

Let $e_I \in K_1 \cup K_2$. Then $e_I \in K_1$ or $e_I \in K_2$. If $e_I \in K_1$ then $e_I = j((e_I, 0))$. If $e_I \in K_2$ then $e_I = j((0, -e_I))$. So $e_I \in \operatorname{Im} j$ in either case. Since the e_I form a free basis, j is surjective as required. □

Theorem (Mayer–Vietoris sequence)**.** Let K_1, K_2 be abstract simplicial complexes in Δ^n. Then there is a long exact sequence

$$H_k(K_1 \cap K_2) \xrightarrow{i_\star} H_k(K_1) \oplus H_k(K_2) \xrightarrow{j_\star} H_k(K_1 \cup K_2) \xrightarrow{\partial_k} H_{k-1}(K_1 \cap K_2) \xrightarrow{i_\star} H_{k-1}(K_1) \oplus H_{k-1}(K_2) \xrightarrow{j_\star} H_{k-1}(K_1 \cup K_2)$$

Proof. Follows from the above theorem and the snake lemma. □

Example. Let K_1, K_2 be abstract simplicial complexes in Δ^n, Δ^m. Then let $K_1 \amalg K_2 \subset \Delta^{n+m+1}$ be the abstract simplicial complex where the vertices of Δ^{n+m+1} are $e_0, \dots, e_n, e'_0, \dots, e'_m$, and we embed K_1 and K_2 into $K_1 \amalg K_2$ in the natural way. More precisely, $e_I \in K_1$ gives $e_I \in K_1 \amalg K_2$, and $e_I \in K_2$ gives $e'_J \in K_1 \amalg K_2$. Then $|K_1 \amalg K_2| = |K_1| \amalg |K_2|$. $K_1 \amalg K_2 = K_1 \cup K'_2$ where K_1, K'_2 are disjoint abstract simplicial complexes in Δ^{n+m+1}, so $K_1 \cap K'_2 = \{e_\varnothing\}$. The Mayer–Vietoris sequence gives

$$H_k(\{e_\varnothing\}) \xrightarrow{i_\star} H_k(K_1) \oplus H_k(K_2) \xrightarrow{j_\star} H_k(K_1 \amalg K_2) \xrightarrow{\partial_k} H_{k-1}(\{e_\varnothing\}) \xrightarrow{i_\star} H_{k-1}(K_1) \oplus H_{k-1}(K_2) \xrightarrow{j_\star} H_{k-1}(K_1 \amalg K_2)$$

Note that $H_k(\{e_\varnothing\}) = 0$. Hence, the sequence

$$0 \longrightarrow H_k(K_1) \oplus H_k(K_2) \longrightarrow H_k(K_1 \amalg K_2) \longrightarrow 0$$

is exact. So $H_k(K_1) \oplus H_k(K_2) \simeq H_k(K_1 \amalg K_2)$.

7.7 Homology of triangulable spaces

Theorem. Let $f_0, f_1 : K \to L$ be simplicial approximations to a continuous map $F : |K| \to |L|$. Then $f_{0\sharp} \sim f_{1\sharp}$, so $f_{0\star} = f_{1\star}$.

Theorem. There is an isomorphism $\nu_K : H_\bullet(BK) \to H_\bullet(K)$ such that $\nu_K = f_\star$ where $f : BK \to K$ is any simplicial approximation to the identity map on $|K|$.

Definition. Let $F : |K| \to |L|$ be continuous. By the simplicial approximation theorem, there exists $f : B^r \to L$ that is a simplicial approximation to F. Define $F_\star : H_\bullet(K) \to H_\bullet(L)$ by $F_\star = f_\star \circ \nu_{K,r}^{-1}$.

Theorem. $F_\star$ is well-defined, so does not depend on the choice of f. $(\mathrm{id}_K)_\star = \mathrm{id}_{H_\bullet(K)}$. Further, $(F \circ G)_\star = F_\star \circ G_\star$.

Theorem. Let $F_0, F_1 : |K| \to |L|$ be continuous with $F_0 \sim F_1$. Then $F_{0\star} = F_{1\star}$.

Proposition. Let $|K| \sim |L|$. Then $H_\bullet(K) \simeq H_\bullet(L)$.

Proof. Let $F : |K| \to |L|$ and $G : |L| \to |K|$ be functions such that $F \circ G \sim \mathrm{id}_{|L|}$ and $G \circ F \sim \mathrm{id}_{|K|}$. Then $F_\star \circ G_\star = \mathrm{id}_{H_\bullet(L)}$ and $G_\star \circ F_\star = \mathrm{id}_{H_\bullet(K)}$ by functoriality. Hence $F_\star$ and $G_\star$ are inverse isomorphisms of groups. □

Definition. A space X is *triangulable* if there exists an abstract simplicial complex K with $|K| \simeq X$.

Remark. The above proposition implies that if X is triangulable, there is a well-defined homology group $H_\bullet(X) = H_\bullet(K)$ where K is any abstract simplicial complex with polyhedron $|K| \simeq X$. Not all topological spaces are homotopy equivalent to a triangulable space. One example is $\bigvee_{i=1}^\infty S^1$.

Proposition. Let $|K|$ be path-connected. Then $H_0(K) \simeq \mathbb{Z}$.

Proof. $C_0(K)$ is generated by the vertices e_i of K. Consider $F_i : \Delta^0 \to |K|$ mapping $e_0 \in \Delta^0$ to $e_i \in K$. Then $H_\star(\Delta^0) = \mathbb{Z} = \langle [e_0] \rangle$, and $F_\star([e_0]) = [e_i]$. Since K is path-connected, $F_i \sim F_j$. So $[e_i] = F_{i\star}([e_0]) = F_{j\star}([e_0]) = [e_j]$. Hence all $[e_i]$ are equal. The $[e_i]$ are not boundaries, so $H_0(K)$ is not trivial. □

Corollary. $H_0(K) = \mathbb{Z}^k$ where k is the number of path-connected components of $|K|$.

Proof. $|K|$ is a disjoint union of the k path-connected components of $|K|$, so $H_0(K)$ is the direct sum of the homology groups of these components. □

We know $S^n \simeq |\mathbb{S}^n|$, so

$$H_k(S^n) = H_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Hence $S^n \sim S^m$ implies $n = m$.

Corollary. $\mathbb{R}^n \simeq \mathbb{R}^m$ implies $n = m$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a homeomorphism. Then $S^{n-1} \sim \mathbb{R}^n \setminus \{0\} \simeq \mathbb{R}^m \setminus \{f(0)\} \sim S^{m-1}$. So $S^{n-1} \sim S^{m-1}$, giving $n = m$. □

Corollary. There is no retraction $r : D^n \to S^{n-1}$.

Proof. We suppose $n > 0$. Let $j : S^{n-1} \to D^n$ be the inclusion. r is a retraction if and only if $r \circ j = \mathrm{id}_{S^{n-1}}$. This gives $(r \circ j)_\star = \mathrm{id}_{H_\cdot(S^{n-1})}$. Note that $H_{n-1}(D^n) = H_{n-1}(\mathbf{\Delta}^n) = 0$, and $H_{n-1}(S^{n-1}) = \mathbb{Z}$. If r is a retraction, then $r_\star$ and $j_\star$ are inverse homomorphisms of groups, but $\mathbb{Z}$ is not isomorphic to 0. So r is not a retraction. □

Theorem (Brouwer fixed point theorem)**.** Let $F : D^n \to D^n$ be a continuous function. Then F has a fixed point.

Remark. This is a generalisation of the intermediate value theorem for high dimensions.

Proof. Suppose there is no fixed point. Then, we define $G : D^n \to S^{n-1}$ by letting $G(x), x, F(x)$ lie in this order on a straight line in D^n. If G is a well-defined continuous map, it is a retraction, contradicting the previous result.

Let $p \in D^n$ and $v \in S^{n-1}$. Let $R_{p,v} = \{p + tv \mid t \geq 0\}$. If $p + tv \in S^{n-1}$, then $\langle p + tv, p + tv\rangle = 1$, so $\langle p, p\rangle + 2t\langle v, p\rangle + t^2 = 1$. Hence

$$t = -\langle p, v\rangle \pm \sqrt{\langle p, v\rangle^2 + 1 - \langle p, p\rangle}$$

We define

$$\tau(p, v) = \max\left(-\langle p, v\rangle \pm \sqrt{\langle p, v\rangle^2 + 1 - \langle p, p\rangle}\right)$$

This is a continuous function. Now, we define $P(p, v) = p + \tau(p, v)v$, which is the intersection of $R_{p,v}$ with S^{n-1}, which is also continuous. So

$$G(x) = P\left(F(x), \frac{x - F(x)}{\|x - F(x)\|}\right)$$

is well-defined and continuous. □

7.8 Homology of orientable surfaces

We can often compute homology groups only using the Mayer–Vietoris sequence and functoriality properties.

Example. Consider the torus T^2. We can write a triangulation K of T^2 as $K_1 \cup K_2$, with $|K_i| \simeq S^1 \times I$, and $|K_1 \cap K_2| \simeq S^1_1 \amalg S^1_2$. Note that the inclusion $\iota_{j,i} : S^1_j \hookrightarrow |K_i|$ is a homotopy equivalence, and $\iota_{1,i} \sim \iota_{2,i}$. Then the Mayer–Vietoris sequence gives

$$\begin{array}{c} H_2(K_1) \oplus H_2(K_2) \longrightarrow H_2(K) \\ \hookrightarrow H_1(K_1 \cap K_2) \xrightarrow[\alpha_1]{} H_1(K_1) \oplus H_1(K_2) \longrightarrow H_1(K) \\ \hookrightarrow H_0(K_1 \cap K_2) \xrightarrow[\alpha_0]{} H_0(K_1) \oplus H_0(K_2) \longrightarrow H_0(K) \longrightarrow 0 \end{array}$$

giving

$$\begin{array}{c} 0 \longrightarrow H_2(K) \\ \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[\alpha_1]{} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(K) \\ \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[\alpha_0]{} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(K) \longrightarrow 0 \end{array}$$

Hence we have short exact sequences

$$0 \longrightarrow H_2(K) \longrightarrow \ker \alpha_1 \longrightarrow 0$$

$$0 \longrightarrow \operatorname{coker} \alpha_1 \longrightarrow H_1(K) \longrightarrow \ker \alpha_0 \longrightarrow 0$$

$$0 \longrightarrow \operatorname{coker} \alpha_0 \longrightarrow H_0(K) \longrightarrow 0$$

The maps α_i are given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore, $\ker \alpha_i \simeq \mathbb{Z}$ and $\operatorname{coker} \alpha_i \simeq \mathbb{Z}$. Hence $H_2(K) \simeq \mathbb{Z}$, $H_1(K) \simeq \mathbb{Z}^2$, and $H_0(K) \simeq \mathbb{Z}$.

$$H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition. Suppose that $0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z}^r \longrightarrow 0$ is exact. Then $B \simeq A \oplus \mathbb{Z}^r$.

Proof. By exactness, $\mathbb{Z}^r \simeq {}^{B}\!/_{A}$. The result then follows from the structure theorem for abelian groups. □

Example. Let L_1 be a triangulation of T^2, and let $L_{1,1}$ be $L_1 \setminus \{\sigma\}$ where σ is a 2-simplex. Then $\partial L_{1,1} \simeq \partial \sigma = \mathbb{S}^1$, and $|L_{1,1}| \sim S^1 \vee S^1$. We inductively define $L_g = L_{g-1,1} \cup_{\mathbb{S}^1} L_{1,1}$, and $L_{g,1} = L_g \setminus \sigma$ where σ is a 2-simplex. Then L_g is a triangulation of the compact surface of genus g. Note also that $L_{g,1} \simeq L_{g-1,1} \cup_{\sigma^1} L_{1,1}$ where σ^1 is an edge of $\mathbb{S}^1$. So $L_{g,1} \sim \bigvee_{i=1}^{2g} S^1$.

Proposition.

$$H_k(L_g) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^{2g} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_k(L_{g,1}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Further, $\iota_{g\star} : H_1(\partial L_{g,1}) \to H_1(L_{g,1})$ is the zero map.

Proof. By induction, we show the result for $H_k(L_g)$ implies the result for $H_k(L_{g,1})$, and then $H_k(L_{g,1})$ gives $H_k(L_{g+1})$. The base case is $H_\bullet(T^2)$ which was shown above. For the first implication, we use the Mayer–Vietoris sequence. Note that $L_g = L_{g,1} \cup_{\partial L_{g,1}} \mathbf{\Delta}^2$. Then,

$$\begin{array}{ccccc} & & H_2(L_{g,1}) \oplus H_2(\mathbf{\Delta}^2) & \longrightarrow & H_2(L_g) \\ \xrightarrow{\partial_2} H_1(\partial L_{g,1}) & \xrightarrow{\iota_1} & H_1(L_{g,1}) \oplus H_1(\mathbf{\Delta}^2) & \longrightarrow & H_1(L_g) \\ \xrightarrow{\partial_1} H_0(\partial L_{g,1}) & \xrightarrow{\iota_0} & H_0(L_{g,1}) \oplus H_0(\mathbf{\Delta}^2) & \longrightarrow & H_0(L_g) \end{array}$$

giving

$$\begin{array}{ccccc} & & 0 \oplus 0 & \longrightarrow & \mathbb{Z} \\ \xrightarrow{\partial_2} \mathbb{Z} & \xrightarrow{\iota_1} & H_1(L_{g,1}) \oplus 0 & \longrightarrow & \mathbb{Z}^{2g} \\ \xrightarrow{\partial_1} \mathbb{Z} & \xrightarrow{\iota_0} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

The bottom row of the Mayer–Vietoris sequence always has this form if $K_1, K_2, K_1 \cap K_2$ are connected. Note that since ι_0 is injective, the map before it is the zero map by exactness, so we can remove the bottom row and replace it with zero. We have that ∂_2 is injective, and $H_1(L_{g,1})$ is torsion-free, so ∂_2 is an isomorphism. Hence ι_1 is the zero map and j is an isomorphism. Since $0 = \iota_1 = \iota_{g\star} + \iota'_\star$, we have $\iota_{g\star} = 0$. Further, as j is an isomorphism, $H_1(L_{g,1}) \simeq H_1(L_g) = \mathbb{Z}^{2g}$ as required.

Now we show the result for $H_k(L_{g,1})$ implies the result for $H_k(L_{g+1})$. Note that $L_{g+1} = L_{g,1} \cup_{\partial L_{g,1}} L_{1,1}$. Hence,

$$\begin{array}{cccccc} & & H_2(L_{g,1}) \oplus H_2(L_{1,1}) & \longrightarrow & H_2(L_{g+1}) & \\ \longrightarrow H_1(\partial L_{g,1}) & \xrightarrow{\iota} & H_1(L_{g,1}) \oplus H_2(L_{1,1}) & \longrightarrow & H_1(L_{g+1}) & \longrightarrow 0 \end{array}$$

so

$$
\begin{array}{c}
0 \oplus 0 \longrightarrow H_2(L_{g+1}) \\
\hookrightarrow \mathbb{Z} \xrightarrow[\iota]{} \mathbb{Z}^{2g} \oplus \mathbb{Z}^2 \longrightarrow H_1(L_{g+1}) \longrightarrow 0
\end{array}
$$

By assumption, ι is the zero map. Hence $H_2(L_{g+1}) \simeq H_1(\partial L_{g,1}) \simeq \mathbb{Z}$ as ∂_2 is an isomorphism. Also, $\mathbb{Z}^{2g+2} \simeq H_1(L_{g+1})$ by exactness. □

7.9 Homology of non-orientable surfaces

Let M_1 be a triangulation of $\mathbb{RP}^2$. Let $M_{r,1}$ be M_r with a 2-simplex removed, so $\partial M_{r,1} \simeq \mathbb{S}^1$. Let $M_{r+1} = M_{r,1} \cup_{\partial M_{r,1}} M_{1,1}$. Then $M_{r+1,1} = M_{r,1} \cup_{\Delta^1} M_{1,1}$, attaching along an interval. For example, $|M_{1,1}|$ is homeomorphic to the Möbius band. Then $M_{r,1} \sim \bigvee_{i=1}^{r} S^1$.

Proposition.

$$
H_k(M_r) = \begin{cases} \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
H_k(M_{r,1}) = \begin{cases} \mathbb{Z}^{r} & k = 1 \\ \mathbb{Z} & k = 0 \\ 0 & \text{otherwise} \end{cases}
$$

Further, $\iota_{r\star} : H_1(\partial M_{r,1}) \to H_1(M_{r,1})$ has the property that $\iota_{r\star}(1)$ is twice a primitive element, or equivalently, $H_1(M_{r,1}) / \operatorname{Im} \iota_{r\star} = \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. We proceed by induction in the same way. For the base case, note that $\partial M_{1,1} \simeq S^1$ and $M_{1,1} \simeq S^1$, and the map from $\partial M_{1,1} \to M_{1,1}$ is given by $z \mapsto z^2$, so the map $H_1(S^1) \to H_1(S^1)$ is given by multiplication by 2. Suppose the result holds for $H_k(M_r)$. Then, $M_r = M_{r,1} \cup_{\partial M_{r,1}} \mathbf{\Delta}^2$, and

$$
\begin{array}{c}
H_2(M_{r,1}) \oplus H_2(\mathbf{\Delta}^2) \longrightarrow H_2(M_r) \\
\hookrightarrow H_1(\partial M_{r,1}) \xrightarrow[\iota_{r\star}]{} H_1(M_{r,1}) \oplus H_1(\mathbf{\Delta}^2) \longrightarrow H_1(M_r) \longrightarrow 0
\end{array}
$$

$\iota_{r\star}$ is injective, so $\partial_2 = 0$, giving $0 \longrightarrow H_2(M_r) \longrightarrow 0$. Hence,

$$
\begin{array}{c}
0 \oplus 0 \longrightarrow 0 \\
\hookrightarrow \mathbb{Z} \xrightarrow[\iota_{r\star}]{} H_1(M_{r,1}) \oplus 0 \longrightarrow \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
\end{array}
$$

Since $H_1(M_{r,1})$ is torsion-free,

$$
0 \longrightarrow \mathbb{Z} \longrightarrow H_1(M_{r,1}) \longrightarrow \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
$$

gives that $H_1(M_{r,1}) = \mathbb{Z}^r$.

Now, $M_{r+1} = M_{r,1} \cup_{\partial M_{r,1}} M_{1,1}$ hence

$$\begin{array}{l} H_2(M_{r,1}) \oplus H_2(M_{1,1}) \longrightarrow H_2(M_{r+1}) \longrightarrow \\ H_1(\mathbb{S}^1) \longrightarrow H_1(M_{r,1}) \oplus H_1(M_{r,1}) \longrightarrow H_1(M_{r+1}) \longrightarrow 0 \end{array}$$

so

$$\begin{array}{l} 0 \oplus 0 \longrightarrow 0 \xrightarrow{0} \\ \mathbb{Z} \longrightarrow H_1(M_{r,1}) \oplus H_1(M_{r,1}) \longrightarrow H_1(M_{r+1}) \longrightarrow 0 \end{array}$$

Hence $H_1(M_{r+1}) \simeq \mathbb{Z}^2 \oplus \mathbb{Z} / (2e_1, 2) \simeq \mathbb{Z}^r \oplus \mathbb{Z} / 2\mathbb{Z}$. □

7.10 Lefschetz fixed point theorem

Let (C, d) be a chain complex over $\mathbb{Q}$ (or any other field). Then $H_\bullet(C)$ is a $\mathbb{Q}$-vector space. Let $f : C \to C$ be a chain map, so it induces $f_\star : H_\bullet(C) \to H_\bullet(C)$. f and $f_\star$ are both linear endomorphisms of vector spaces.

Definition. The *Lefschetz number* of f is $L(f) = \sum_k (-1)^k \operatorname{tr} f_k$ where $f_k : C_k \to C_k$, and $L(f_\star) = \sum_k (-1)^k \operatorname{tr} f_{k\star}$ where $f_{k\star} : H_k(C) \to H_k(C)$.

Proposition. $L(f) = L(f_\star)$.

Proof. Let $U_k = \operatorname{Im} d_{k+1} \subseteq \ker d_k \subseteq C_k$. Then, $\ker d_k = U_k \oplus V_k$, and $C_k = U_k \oplus V_k \oplus U'_k$. Then $d : U'_k \to U_{k-1}$ is an isomorphism. With respect to this decomposition, d is a matrix in block form given by

$$d = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also, $f(\operatorname{Im} d_{k+1}) \subseteq \operatorname{Im} d_{k+1}$ since f is a chain map, and $f(\ker d_k) \subseteq \ker d_k$. So in block form,

$$f = \begin{pmatrix} A_k & X_k & \star \\ 0 & B_k & \star \\ 0 & 0 & A'_k \end{pmatrix}$$

Then, the equation $df = fd$ shows $A_k = A'_{k+1}$. Hence, $H_k(C) = \ker d_k / \operatorname{Im} d_{k+1} = U_k \oplus V_k / U_k \simeq V_k$, and $f_{k\star} : H_k(C) \to H_k(C)$ maps $[v]$ to $[B_k v + X_k v] = [B_k v]$, so $f_{k\star}$ is multiplication by B_k. Then $L(f) = \sum_k (-1)^k \operatorname{tr} f_k = \sum (-1)^k (\operatorname{tr} A_k + \operatorname{tr} B_k + \operatorname{tr} A_{k-1}) = \sum (-1)^k \operatorname{tr} B_k = L(f_\star)$. □

Definition. Let $C = C_\bullet(K)$. Then the *Euler characteristic* is defined by $\chi(C) = L(\mathrm{id}_C)$. Hence $\chi(C(K)) = \sum_k (-1)^k \dim C_k(K)$. Note that $L(\mathrm{id}_C) = L(\mathrm{id}_{H_\bullet(K)}) = \sum_k (-1)^k \dim H_k(K)$ depends only on $|K|$.

Theorem (Lefschetz fixed point theorem)**.** Let $F : |K| \to |K|$ be a continuous map. Let $L(F) = L(F_\star)$ be the Lefschetz number of F, where $F_\star : H_\bullet(K) \to H_\bullet(K)$. Then if $L(F) \neq 0$, F has a fixed point.

Remark. This is a generalisation of the Brouwer fixed point theorem.

Proof sketch. If F has no fixed point, then since $|K|$ is compact, there exists $\varepsilon > 0$ such that $|F(x) - x| \geq \varepsilon$ for all x. If $f : B^{r+n}K \to B^r K$ is a simplicial approximation of F, then the above implies that $F_\star(\sigma)$ does not contain σ for any simplex $\sigma \in C_\bullet(K)$. Hence $L(F) = L(f) = 0$. □