Galois Theory

Cambridge University Mathematical Tripos: Part II

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1 Polynomials

1.1 Introduction

Galois theory concerns itself with solving polynomial equations of higher degree, and discussing how the symmetries of these polynomials relate to their solubility. The modern interpretation of Galois theory is more interested in the fields that particular polynomials generate, rather than their particular solutions; this naturally extends to studying symmetries of fields.

1.2 Solving quadratics, cubics and quartics

Methods for solving quadratic equations have been known since the time of the Babylonians. Consider $aX^2 + bX + c$, and complete the square into $\left(X + \frac{1}{2}b\right)^2 + c - \frac{b^2}{4}$. This leads directly into the usual formula.

Alternatively, consider $(X-x_1)(X-x_2)$ and expand, giving $X^2-(x_1+x_2)X+x_1x_2$. Thus, $x_1+x_2 = -b$ and $x_1x_2 = c$. We can write $x_1 = \frac{1}{2}[(x_1 + x_2) + (x_1 - x_2)]$, where $x_1 + x_2 = b$ and $(x_1 - x_2)^2 = b^2 - 4c$.

Cubics were solved much later, in the early 16th century, by the Italian mathematician del Ferro. Consider the cubic X^3+aX^2+bX+c, written as $(X-x_1)(X-x_2)(X-x_3)$. Multiplying, we find

$$x_1 + x_2 + x_3 = -a; \quad x_1x_2 + x_2x_3 + x_3x_1 = b; \quad x_1x_2x_3 = -c$$

Without loss of generality we can set $a = 0$ by replacing $X \mapsto X - \frac{a}{3}$. Now,

$$x_1 = \frac{1}{3}\left[(x_1 + x_2 + x_3) + \underbrace{(x_1 + \omega x_2 + \omega^2 x_3)}_{u} + \underbrace{(x_1 + \omega^2 x_2 + \omega x_3)}_{v}\right]$$

where $\omega = e^{\frac{2\pi i}{3}}$. The u, v are known as Lagrange resolvents. Applying a cyclic permutation to x_1, x_2, x_3 in u or v, we find $u \mapsto \omega u$ and $v \mapsto \omega v$. Hence, the cubes of u and v are invariant under cyclic permutations of x_1, x_2, x_3. Under a permutation $x_2 \mapsto x_3, x_3 \mapsto x_2$, u and v swap. Hence, $u^3 + v^3$ and u^3v^3 are invariant under all permutations of roots. A general fact that we will prove later is that such invariant expressions can be written in terms of the coefficients of the polynomial. In this case, we have

$$u^3 + v^3 = -27c; \quad u^3v^3 = -27b^2$$

Now, u^3 and v^3 are the roots of the quadratic $Y^2 + 27cY - 27b^2$. This then provides a formula for the root x_1. This process is known as Cardano's formula.

Similarly, the quartic $X^4+aX^3+bX^2+cX+d$ can be solved by producing an auxiliary cubic equation, in a similar way to the auxiliary quadratic equation found for the cubic case above. However, the same process does not work for the quintic; the auxiliary equation has a degree which is too large. The underlying reason behind this is to do with group theory, and in particular, the group structure of S_5 and A_5. This will be explored later in the course.

1.3 Polynomial rings

In this course, *ring* means a commutative nonzero ring. If R is a ring, $R[X]$ denotes the ring of polynomials with elements $\sum_{i=0}^{n} a_i X^i$, and the usual operations of addition and multiplication. A polynomial $f \in R[X]$ can be interpreted as a function $f : R \to R$, given by $x \mapsto \sum_{i=0}^{n} a_i x^i$. It is, however,

important to distinguish the polynomial and its associated function; the polynomial is not in general uniquely determined by the function. For example, let $R = \mathbb{Z}/p\mathbb{Z}$, so for all $a \in R$, we have $a^p = a$, and hence X^p and X are different polynomials yet represent the same function.

Recall from Groups, Rings and Modules that if $R = K$ is a field, $K[X]$ is a Euclidean domain (and hence is a unique factorisation domain, a Noetherian ring, a principal ideal domain, and an integral domain). Hence, there is a division algorithm: for polynomials $f, g \in K[X]$, there exists a unique $q, r \in K[X]$ such that $f = gq + r$ and $\deg r < \deg g$, where we denote $\deg 0 = -\infty$. If $g = X - a$ is linear, $f = (X-a)q + r$ where $r = f(a) \in K$; this is the familiar remainder theorem. Note that every polynomial $f \in K[X]$ is a product of irreducible polynomials since $K[X]$ is a unique factorisation domain, and there are greatest common divisors which can be computed using Euclid's algorithm in the usual way.

Proposition. Let K be a field, and $0 \neq f \in K[X]$. Then, f has at most $\deg f$ roots in K.

Proof. If f has no roots, the proof is complete. If f has a root a, consider $f = (X-a)q + f(0) = (X-a)q$. For a root b of f, either $b = a$ or $q(b) = 0$. By induction, q has at most $\deg q$ roots, since $\deg q < \deg f$. Then $\deg q + 1 \leq \deg f$ as required. □

1.4 Symmetric polynomials

Definition. Let R be a ring, and let $n \geq 1$. A polynomial $f \in R[X_1, \dots, X_n]$ is *symmetric* if, for every permutation $\sigma \in S_n$, we have $f(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = f(X_1, \dots, X_n)$, where S_n is the symmetric group of degree n.

Note that constant polynomials are symmetric, and the property of symmetry is closed under addition and multiplication. Hence, the set of symmetric polynomials is a subring of $R[X_1, \dots, X_n]$.

Example. $X_1 + \cdots + X_n$ is symmetric. More generally, $p_k = X_1^k + \cdots + X_n^k$ is symmetric.

Proposition. Let $f\sigma(X) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)})$. This gives an action (on the right) of the group S_n on $R[X_1, \dots, X_n]$. A polynomial $f \in R[X_1, \dots, X_n]$ is symmetric if and only if f is fixed under the action of S_n; in other words, $f\sigma = f$ for all $\sigma \in S_n$.

Definition. The *elementary symmetric functions* or *elementary symmetric polynomials* are

$$s_r(X_1, \dots, X_n) = \sum_{i_1 < \cdots < i_r} X_{i_1} X_{i_2} \cdots X_{i_r}$$

For instance,

$$s_2(X_1, X_2, X_3) = X_1X_2 + X_1X_3 + X_2X_3$$

It is clear that these are symmetric polynomials.

Definition. A *monomial* is an expression of the form $X_I = X_1^{I_1} \cdots X_n^{I_n}$ for $I \in \mathbb{N}^n$. The *(total) degree* of a monomial is $\sum_{i=1}^n I_i$. A *term* is a scalar multiple of a monomial. A polynomial is uniquely characterised by a sum of terms. The total degree of a polynomial is the maximum total degree of its terms.
Monomials are equipped with a *lexicographic ordering*, where we say monomials $X_I > X_J$ if either $I_1 > J_1$ or $I_1 = J_1$ and for some $r \in \{1, \dots, n-1\}$ we have $I_1 = J_1, \dots, I_r = J_r, I_{r+1} > J_{r+1}$. This is a total order.

Theorem. Every symmetric polynomial in n variables over a ring R can be expressed as a polynomial in the s_r for $1 \leq r \leq n$, with coefficients in R. Further, there are no non-trivial relations between the s_r.

Remark. Consider the ring homomorphism $\theta : R[Y_1, \dots, Y_n] \to R[X_1, \dots, X_n]$ given by $\theta(Y_r) = s_r$ and $\theta(r) = r$ for $r \in R$. The first part of the above theorem stipulates that $\operatorname{Im}\theta$ is the set of symmetric polynomials. The second part implies that θ is injective, since any element of $\ker\theta$ is a polynomial between the s_r that evaluates to zero.

Note that we can equivalently define the s_r as

$$\prod_{i=1}^{n}(T + X_i) = T^n + s_1 T^{n-1} + \cdots + s_{n-1}T + s_n$$

If we need to specify the number of variables, we use $s_{r,n}$ instead of s_r.

Proof. Let d be the total degree of a symmetric polynomial f. Let X_I be the largest (in lexicographic order) monomial which occurs in f with coefficient c. Since f is symmetric, any permutation of the X_i yields another monomial that occurs in f. Hence, $I_1 \geq I_2 \geq \cdots \geq I_n$, because otherwise the rearranged monomial that satisfies this will be a strictly larger monomial in f. We can therefore write

$$X_I = X_1^{I_1 - I_2}(X_1X_2)^{I_2 - I_3} \cdots (X_1 \dots X_n)^{I_n}$$

Consider

$$g = s_1^{I_1 - I_2} s_2^{I_2 - I_3} \cdots s_n^{I_n}$$

By construction, the largest monomial in g is X_I. Since g is symmetric, cg is symmetric. By induction, we may assume $f - cg$ is expressible as a sum of symmetric polynomials as it has total degree no larger than d, its leading monomial term is smaller than X_I, and there are only finitely many monomials of degree at most d. Hence f is also expressible as a sum of polynomials as required.

Now we prove uniqueness by induction on n. Let $G \in R[Y_1, \dots, Y_n]$ such that $G(s_{1,n}, \dots, s_{n,n}) = 0$. We want to show that G is the zero polynomial. If $n = 1$, the result is trivial as $s_{1,1} = X_1$. If $G = Y_n^k H$ with Y_n not dividing H, then $s_{n,n}^k H(s_{1,n}, \dots, s_{n,n}) = 0$. Since $s_{n,n} = X_1 \dots X_n$, it is not a zero divisor in $R[X_1, \dots, X_n]$. Hence $H(s_{1,n}, \dots, s_{n,n}) = 0$. Without loss of generality, we can assume that G is not divisible by Y_n. Now, replacing X_n with zero, $s_{k,n}$ is mapped to $s_{k,n-1}$ for $k \neq n$, and $s_{n,n}$ is mapped to zero. Hence, $G(s_{1,n-1}, \dots, s_{n-1,n-1}, 0) = 0$. By induction, $G(Y_1, \dots, Y_{n-1}, 0) = 0$. Hence $Y_n \mid G$, contradicting our assumption. □

Example. Consider, for $n \geq 3$,

$$f = \sum_{i \neq j} X_i^2 X_j$$

The leading term is $X_1^2X_2 = X_1(X_1X_2)$, so we consider

$$\begin{aligned} f - s_1s_2 &= \left(\sum_{i\neq j} X_i^2X_j\right) - \sum_i \sum_{j<k} X_iX_jX_k \\ &= \left(\sum_{i\neq j} X_i^2X_j\right) - \left(\sum_{i\neq j} X_i^2X_j + 3\sum_{i<j<k} X_iX_jX_k\right) \\ &= -3\sum_{i<j<k} X_iX_jX_k \\ &= -3s_3 \end{aligned}$$

Hence $f = s_1s_2 - 3s_3$.

Consider $f = p_5 = \sum_i X_i^5$. Computing this in terms of elementary symmetric polynomials by hand is somewhat tedious, but there are various results, such as Newton's formulae, which can help in simplifying such expressions.

Theorem (Newton's formulae)**.** Let $n \geq 1$. Then for all $k \geq 1$,

$$p_k - s_1p_{k-1} + \cdots + (-1)^{k-1}s_{k-1}p_1 + (-1)^k ks_k = 0$$

By convention, let $s_0 = 1$ and $s_r = 0$ if $r > n$.

Proof. It suffices to consider $R = \mathbb{Z}$ (or, for example, $R = \mathbb{R}$). Consider the generating function

$$F(T) = \prod_{i=1}^n (1 - X_iT) = \sum_{r=0}^n (-1)^r s_r T^r$$

Note that for polynomials $f(x), g(x)$, their formal derivatives satisfy

$$\frac{\frac{\mathrm{d}}{\mathrm{d}T}(fg)}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}$$

Then, taking the logarithmic derivative with respect to T,

$$\begin{aligned} \frac{F'(T)}{F(T)} &= \frac{\frac{\mathrm{d}}{\mathrm{d}T}\prod_{i=1}^n (1 - X_iT)}{\prod_{i=1}^n (1 - X_iT)} \\ &= \sum_{i=1}^n \frac{\frac{\mathrm{d}}{\mathrm{d}T}(1 - X_iT)}{1 - X_iT} \\ &= -\sum_{i=1}^n \frac{X_i}{1 - X_iT} \\ &= \frac{-1}{T}\sum_{i=1}^n \sum_{r=1}^\infty X_i^r T^r \\ &= \frac{-1}{T}\sum_{r=1}^\infty p_r T^r \end{aligned}$$

Hence,

$$-TF'(T) = s_1T - 2s_2T^2 + \cdots + (-1)^{n-1}ns_nT^n$$

but also

$$-TF'(T) = F(T)\sum_{r=1}^{\infty} p_rT_r = (s_0 - s_1T + \cdots + (-1)^n s_nT^n)\big(p_1T + p_2T^2 + \dots\big)$$

Equating the coefficients of powers of T, we find the identity as required by the theorem. □

Example. The *discriminant polynomial* is

$$D(X_1, \dots, X_n) = \Delta(X_1, \dots, X_n)^2$$

where

$$\Delta(X_1, \dots, X_n) = \prod_{i<j}(X_i - X_j)$$

This is used in defining the sign of a permutation: applying a permutation σ to Δ multiplies Δ by the sign of σ. Hence D is symmetric. Therefore, D can be written in terms of the symmetric polynomials.

$$D(X_1, \dots, X_n) = d(s_1, \dots, s_n)$$

where d has integer coefficients. For example, $n = 2$ gives $D = (X_1 - X_2)^2 = s_1^2 - 4s_2$.

Definition. Let $f = T^n + \sum_{i=0}^{n-1} a_{n-i}T^i$ be a monic polynomial in $R[T]$. Its discriminant is

$$\mathrm{Disc}(f) = d(-a_1, a_2, -a_3, \dots, (-1)^n a_n) \in R$$

Observe that if f is a product of linear polynomials $f = \prod_{i=1}^{n}(T - x_i)$, then

$$a_r = (-1)^r s_r(x_1, \dots, x_n)$$

giving

$$\mathrm{Disc}(f) = \prod_{i<j}(x_i - x_j)^2 = D(x_1, \dots, x_n)$$

In particular, if $R = K$ is a field, $\mathrm{Disc}(f) = 0$ if and only if f has a repeated root. For example, $\mathrm{Disc}(T^2 + bT + c) = b^2 - 4c$.

2 Fields

2.1 Definition

Definition. A *field* is a commutative nonzero ring K with a 1, in which every nonzero element is invertible. The set of nonzero elements in K is therefore a group under multiplication, known as the multiplicative group of K, denoted $K^\times$.

Definition. The *characteristic* of a field K is the least positive integer p such that $p \cdot 1 = 0$; or if such an integer does not exist, its characteristic is zero.

Example. $\mathbb{Q}$ has characteristic zero. $\mathbb{F}_p = {}^{\mathbb{Z}}\!/_{p\mathbb{Z}}$ has characteristic p, when p is prime.

Remark. The characteristic of a field is always prime or zero.

Definition. The *prime subfield* of a field K is the smallest subfield of K, which is isomorphic to $\mathbb{F}_p$ (if its characteristic is a prime p) or $\mathbb{Q}$ (if its characteristic is zero).

Proposition. Let $\varphi : K \to L$ be a field homomorphism. Then φ is an injection.

Proof. We have $\varphi(1_K) = 1_L \neq 0_L$ by the definition of a ring homomorphism. Then $\ker \varphi$ is a proper ideal of K. But the only proper ideal of a field is the zero ideal, so $\ker \varphi = (0)$. □

2.2 Field extensions

Definition. Let $K \subset L$ be fields (implicitly assuming that the field operations and identity elements on K and L are the same). We say K is a subfield of L, and L is a *field extension* of K, denoted L/K (read 'L over K'). If $i : K \to L$ is a field homomorphism, we say that i is an isomorphism of K with the subfield $i(K) \subset L$; in this case, we identify K with $i(K)$ and say L is a field extension of K.

Remark. The notation L/K is not related to quotients or division.

Example. (i) $\mathbb{C}/\mathbb{R}/\mathbb{Q}$.

(ii) $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}/\mathbb{Q}$.

Definition. Let $K \subset L$, and $x \in L$. We define $K[x] = \{p(x) \mid p \in K[T]\}$, the ring of polynomial expressions in x. This is a subring of L, but is not in general a field. We further define $K(x) = \left\{ \frac{p(x)}{q(x)} \,\middle|\, p, q \in K[T], q(x) \neq 0 \right\}$ to be the field of fractions of $K[x]$, which is the field of rational expressions in x. This is a subfield of L, usually read 'K adjoin x'. For $x_1, \dots, x_n \in L$, we define

$$K[x_1, \dots, x_n] = \{p(x_1, \dots, x_n) \mid p \in K[T_1, \dots, T_n]\}$$
$$K(x_1, \dots, x_n) = \left\{ \frac{p(x_1, \dots, x_n)}{q(x_1, \dots, x_n)} \,\middle|\, p, q \in K[T_1, \dots, T_n], q(x_1, \dots, x_n) \neq 0 \right\}$$

Remark. One can check that $K(x_1, \dots, x_{n-1})(x_n) = K(x_1, \dots, x_n)$ and similarly for $K[x_1, \dots, x_n]$.

2.3 Field extensions as vector spaces

Remark. A field extension L/K turns L into a K-vector space by forgetting the multiplication between elements of L.

Definition. A field extension L/K is called a *finite extension* if L is a finite-dimensional K-vector space. In this case, we write $[L : K] = \dim_K L$ for the dimension of this vector space, known as the *degree* of the extension. Otherwise, we say L/K is an *infinite extension*, and write $[L : K] = \infty$.

Remark. $[L : L] = \dim_L L = 1$. As a K-vector space, $L \cong K^{[L:K]}$.

Example. $\mathbb{C}/\mathbb{R}$ is a finite extension of degree two.

If K is any field, the extension $K(X)/K$ is an infinite extension, where $K(X)$ is the field of rational functions, the field of fractions of the polynomial ring $K[X]$. This is because $1, X, X^2, \ldots$ are linearly independent.

$\mathbb{R}/\mathbb{Q}$ is an infinite extension. This follows by a countability argument. If $\mathbb{R}/\mathbb{Q}$ were a finite extension of degree n, we would have $\mathbb{R} \cong \mathbb{Q}^n$, but the left hand side is uncountable and the right hand side is countable.

This course is largely concerned with properties and symmetries of finite field extensions.

Definition. An extension is *quadratic*, *cubic*, etc. if its degree is 2, 3, etc.

Proposition. Suppose K is a finite field (necessarily of characteristic p for $p \neq 0$ a prime). Then $|K|$ is a power of p.

Proof. Note that $K/\mathbb{F}_p$ is a finite extension, and so $K \cong \mathbb{F}_p^n$, giving $|K| = p^n$. □

We will later show that for all prime powers $q = p^n$, there exists a finite field $\mathbb{F}_q$ with q elements.

Theorem (tower law)**.** Let M/L, L/K be a pair of field extensions. Then M/K is a finite extension if and only if M/L and L/K are finite. If so, we have $[M : L][L : K] = [M : K]$.

It is easier to prove a more general statement.

Theorem. Let L/K and V is an L-vector space. Then V is a K-vector space, and $\dim_K V = [L : K] \dim_L V$ (with the obvious meaning if any of these expressions are infinite).

Taking $V = M$ proves the tower law as required.

Proof. Let $\dim_L V = d < \infty$. Then $V \cong L \oplus \cdots \oplus L = L^d$ as an L-vector space, so this also holds as a K-vector space. But since $L \cong K^{[L:K]}$ as a K-vector space, we have $V \cong (K^{[L:K]})^d \cong K^{d[L:K]}$ as a K-vector space.

If V is finite-dimensional over K, then a K-basis for V will span V over L, so V is finite-dimensional over L. Thus if V is infinite-dimensional over L, it is infinite-dimensional over K.

Likewise, if $[L : K] = \infty$ and $V \neq 0$, then V has an infinite set of linearly independent vectors as a K-vector space, so $\dim_K V = \infty$. □

Proposition. Let K be a field, and $G \subset K^\times$ be a finite subgroup of the multiplicative group. Then G is cyclic. In particular, if K is finite, $K^\times$ is cyclic.

Proof. We can find m_i such that

$$G \cong \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}$$

where $1 < m_1 \mid m_2 \mid \cdots \mid m_k = m$ by the structure theorem for abelian groups. Then, every element of G satisfies $x^m = 1$. Since K is a field, the polynomial $T^m - 1$ has at most m roots. Every element of G is a root of this polynomial, so $|G| \leq m$. This can only happen when $k = 1$, so $G = \mathbb{Z}/m\mathbb{Z}$. □

Remark. If $K = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, there exists $a \in \{1, \dots, p-1\}$ such that $\mathbb{Z}/p\mathbb{Z} = \{1, a, a^2, \dots, a^{p-1}\}$. Such an a is called a *primitive root* mod p.

Proposition. Let R be a ring, p be a prime such that $p \cdot 1_R = 0_R$ (for instance, R could be a field of characteristic p). Then, the map $\varphi_p : R \to R$ given by $\varphi_p(x) = x^p$ is a homomorphism, known as the *Frobenius endomorphism*.

Proof. First, $\varphi_p(1) = 1^p = 1$ and $\varphi_p(x)\varphi_p(y) = x^p y^p = (xy)^p = \varphi_p(xy)$. For $x, y \in R$,

$$\begin{aligned}\varphi_p(x+y) &= \binom{p}{0}x^p y^0 + \binom{p}{1}x^{p-1}y^1 + \cdots + \binom{p}{p-1}x^1 y^{p-1} + \binom{p}{p}x^0 y^p \\ &= x^p + y^p = \varphi_p(x) + \varphi_p(y)\end{aligned}$$

since $p \mid \binom{p}{k}$ for $k \in \{1, \dots, p-1\}$ by primality of p. □

Example. This gives another proof of Fermat's little theorem $x^p \equiv x \bmod p$, by induction on x noting that $(x+1)^p \equiv x^p + 1 \bmod p$.

2.4 Algebraic elements and minimal polynomials

Definition. Let L/K be an extension and $x \in L$. x is *algebraic over K* if there exists a nonzero polynomial $f \in K[T]$ such that $f(x) = 0$. Otherwise, we say x is *transcendental over K*.

For $f \in K[T]$, we have $f(x) \in L$. Varying f, this gives a map $\mathrm{ev}_x : K[T] \to L$ defined by $f \mapsto f(x)$. This is a ring homomorphism.

The kernel $I = \ker(\mathrm{ev}_x) \subset K[T]$ is an ideal, the set of polynomials which vanish at x. As $\mathrm{Im}(\mathrm{ev}_x)$ is a subring of L which is a field, it is an integral domain. In particular, I is a prime ideal, so either $I = 0$, in which case x is transcendental over K, or there exists a unique monic irreducible polynomial $0 \neq g \in K[T]$ such that $I = (g)$, in which case x is algebraic and we say g is the *minimal polynomial* of x over K. In this case, $f(x) = 0$ if and only if $g \mid f$. We write $m_{x,K}$ for the minimal polynomial of x over K. Note that $m_{x,K}$ is the monic polynomial in K of least degree with x as a root.

Example. If $x \in K$, $m_{x,K} = T - x$. If p is prime and $d \geq 1$, $T^d - p \in \mathbb{Q}[T]$ is irreducible by Eisenstein's criterion, so it is the minimal polynomial of $\sqrt[d]{p} \in \mathbb{R}$ over $\mathbb{Q}$. If p is prime, $z = e^{\frac{2\pi i}{p}}$ is a root of $T^p - 1 = (T-1)(T^{p-1} + T^{p-2} + \cdots + 1) = (T-1)g(T)$. Note that

$$g(T+1) = \binom{p}{p}T^{p-1} + \binom{p}{p-1}T^{p-2} + \cdots + \binom{p}{2}T + \binom{p}{1}$$

This is irreducible by Eisenstein's criterion, so g is minimal for z over $\mathbb{Q}$.

We say the degree of an algebraic element x over K is the degree of its minimal polynomial, written $\deg_K x = \deg(x/K)$.

Proposition. Let L/K and $x \in L$. Then, the following are equivalent.
(i) x is algebraic over K.
(ii) $[K(x) : K]$ is finite.
(iii) $K[x]$ is finite-dimensional as a K-vector space.
(iv) $K[x] = K(x)$.
(v) $K[x]$ is a field.
If these hold, $\deg x = [K(x) : K]$.

Proof. (ii) implies (iii). This follows since $K[x] \subseteq K(x)$.

(iv) is equivalent to (v) is trivial.

(iii) implies (v) and (ii). Let $0 \neq y = g(x) \in K[x]$. Consider the map $K[x] \to K[x]$ given by $z \mapsto yz$. This is a K-linear transformation, and since $y \neq 0$ this is injective. Because $\dim K[x]$ is finite, this injective map must be a bijection. Therefore there exists z such that $yz = 1$, so y is invertible. Hence (v) holds. Since (v) implies (iv), $[K(x) : K] = \dim_K K[x] < \infty$ as required for (ii).

(v) implies (i). If $x = 0$, the proof is complete, so assume $x \neq 0$. Then $x^{-1} = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$. Therefore, $a_n x^{n+1} + \cdots + a_0 x - 1 = 0$, so x is algebraic over K.

(i) implies (v), (iii), and the degree formula. The image of $\mathrm{ev}_x : K[T] \to L$ is the subring $K[x] \subset L$. If x is algebraic over K, $\ker(\mathrm{ev}_x) = (m_{x,K})$ is a maximal ideal by irreducibility of $m_{x,K}$. By the first isomorphism theorem, $K[T]/(m_{x,K}) \cong K[x]$. But quotients by maximal ideals are fields, so $K[x]$ is a field, proving (v). This polynomial is monic of degree $d = \deg_K x$. Hence $K[T]/(m_{x,K})$ has a K-basis $1, T, \ldots, T^{d-1}$. Thus, $\dim_K K[x] = d = [K(x) : K] < \infty$, proving (iii) and the degree formula. □

Corollary. $x_1, \ldots, x_n$ are algebraic over K if and only if $L = K(x_1, \ldots, x_n)$ is finite over K. If so, every element of $K(x_1, \ldots, x_n)$ is algebraic over K.
If x, y are algebraic over K, then so are $x \pm y$, xy, and x^{-1} if x is nonzero. If L/K is a field extension, the set of algebraic elements of L forms a subfield of L.

Proof. If x_n is algebraic over K, then it is also algebraic over $K(x_1, \ldots, x_{n-1})$. Hence the extension $L/K(x_1, \ldots, x_{n-1})$ is finite. By induction on n, the tower law gives the required result. Conversely, if L is finite over K, the subfield $K(y)$ is finite over K for all $y \in L$, so y is algebraic over K.

Suppose x, y are algebraic over K. Then $x \pm y, xy, x^{-1} \in K(x, y)$, which is finite over K as required. □

Example. Consider $z = e^{2\pi i/p} \in \mathbb{C}$ where p is an odd prime. This has degree $p-1$ as discussed above. Now consider $x = 2\cos\frac{2\pi}{p}$, so $x = z + \frac{1}{z} \in \mathbb{Q}(z)$. This is algebraic over $\mathbb{Q}$ because it belongs to this finite extension. Note that $\mathbb{Q}(z) \supset \mathbb{Q}(x) \supset \mathbb{Q}$, and $z^2 - xz + 1 = 0$. Hence the degree of z over $\mathbb{Q}(x)$ is at most 2. But $[\mathbb{Q}(z) : \mathbb{Q}(x)] \neq 1$ because $z \in \mathbb{C} \setminus \mathbb{R}$. By the tower law, we must have $[\mathbb{Q}(z) : \mathbb{Q}] = \frac{p-1}{2}$.

We can now derive the minimal polynomial by considering $z^{\frac{p-1}{2}} + z^{\frac{p-3}{2}} + \cdots + z^{\frac{-p-1}{2}} = 0$. Since $z + z^{-1} = x$, we can express this as a polynomial in x of degree $\frac{p-1}{2}$.

Example. Let $x = \sqrt{m} + \sqrt{n}$ where m, n are integers, and m, n, mn are not squares. We know that $n = (x - \sqrt{m})^2 = x^2 - 2x\sqrt{m} + m$, so $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{m})] \leq 2$. By symmetry, $[\mathbb{Q}(x) : \mathbb{Q}(\sqrt{n})] \leq 2$. Note that $\sqrt{m} \in \mathbb{Q}(x)$ because $\frac{x^2+m-n}{2x} = \sqrt{m}$.

m, n are not squares, so $[\mathbb{Q}(\sqrt{m}) : \mathbb{Q}] = 2$. By the tower law we have $[\mathbb{Q}(x) : \mathbb{Q}] \in \{2, 4\}$. If $[\mathbb{Q}(x) : \mathbb{Q}] = 2$, we have $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{n})$. In this case, $\sqrt{m} = a + b\sqrt{n} \implies m = a^2 + b^2 n + 2ab\sqrt{n}$, but n is not a square, so by rationality, $ab = 0$. But if $b = 0$, m is a square, and if $a = 0$, $mn = b^2 n^2$ is a square. Hence the degree of the field extension is 4.

Definition. An extension L/K is *algebraic* if all elements of L are algebraic over K.

Lemma. Let $M/L/K$, where L/K is algebraic. Suppose x is algebraic over L. Then x is algebraic over K.

Proof. There exists $f = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in L[T]$ where $f \neq 0$ and $f(x) = 0$. Let $L_0 = K(a_0, \ldots, a_{n-1})$. As each $a_i \in L$ is algebraic over K, we must have that $[L_0 : K]$ is finite. As $f \in L_0[T]$, x is algebraic over L_0. So $[L_0(x) : L_0] < \infty \implies [L_0(x) : K] < \infty$. Hence $[K(x) : K] < \infty$, so x is algebraic over K. □

Proposition. (i) Finite extensions are algebraic.
(ii) $K(x)$ is algebraic over K if and only if x is algebraic over K.
(iii) If $M/L/K$, we have M/K is algebraic if and only if M/L and L/K are algebraic.

Proof. (i) $[L : K] < \infty$, so for all $x \in L$, $[K(x) : K] < \infty$, so x is algebraic.

(ii) Certainly if $K(x)$ is algebraic over K, we have that x is algebraic over K. Conversely, if x is algebraic over K, $[K(x) : K]$ is finite, so it is algebraic by part (i).

(iii) Suppose M/K is algebraic. Then for all $x \in M$, we have that x is algebraic over K, so it satisfies a polynomial $f \in K[T]$. Hence $f \in L[T]$ is another polynomial that x satisfies, so M/L is algebraic. L/K is clearly algebraic because it is contained within M.

Conversely, suppose M/L and L/K are algebraic. Let $x \in M$. Then by the previous lemma, x is algebraic over K as required.

□

Example. Let $K = \mathbb{Q}$ and $L = \{x \in \mathbb{C} \mid x \text{ is algebraic over } \mathbb{Q}\} = \overline{\mathbb{Q}}$. This extension $\overline{\mathbb{Q}}/\mathbb{Q}$ is algebraic, but not finite. Indeed, for every $n \geq 1$, $\sqrt[n]{2} \in L$, and $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$ by irreducibility of $T^n - 2$. In particular, L contains subfields of arbitrarily large degree, so cannot be a finite extension.

2.5 Algebraic numbers in the real line and complex plane

Traditionally, we call $x \in \mathbb{C}$ algebraic if it is algebraic over $\mathbb{Q}$, otherwise it is transcendental. $\overline{\mathbb{Q}} = \{x \mid x \text{ algebraic}\}$ is a proper subfield of $\mathbb{C}$. Indeed, $\mathbb{Q}[T]$ is a countable set, and $\mathbb{C}$ is uncountable. However, it is difficult to explicitly find an element of $\mathbb{C} \setminus \overline{\mathbb{Q}}$, or to show that a given number is transcendental.

Example. Liouville's constant $c = \sum_{n \geq 1} 10^{-n!}$ is transcendental, as proven in IA Numbers and Sets. This can be proven by showing that algebraic numbers cannot be 'well approximated' by rationals.

Example. Hermite and Lindemann showed that e and π are transcendental.

Example. Let x, y be algebraic, and $x \neq 0, 1$. Gelfond and Schneider showed that x^y is algebraic if and only if y is rational. In particular, $e^\pi = (-1)^{-i}$ is transcendental.

2.6 Ruler and compass constructions

Definition. A *ruler and compass construction* in plane geometry is a drawing constructed with the following methods.

(i) Given P_1, P_2, Q_1, Q_2 in the plane and $P_i \neq Q_i$, we can construct the point of intersection of the lines P_1Q_1 and P_2Q_2, if indeed they do intersect.
(ii) Given P_1, P_2, Q_1, Q_2 in the plane and $P_i \neq Q_i$, we can construct the points of intersection of the circles with centres P_i that pass through the Q_i, if they intersect.
(iii) Similarly we can construct the points of intersection of a line and a circle.

A point $(x, y) \in \mathbb{R}^2$ is *constructible* from a set $\{(x_1, y_1), \dots, (x_n, y_n)\}$ if it can be obtained by finitely many expansions of the set under applications of the above operations. A real number $x \in \mathbb{R}$ is *constructible* if $(x, 0)$ is constructible from $\{(0, 0), (1, 0)\}$.

Remark. Every rational is constructible. Square roots of constructible numbers are constructible.

Definition. Let $K \subseteq \mathbb{R}$ be a subfield of the reals. We say K is *constructible* if there exists $n \in \mathbb{N}$ and fields $\mathbb{Q} = F_0 \subset F_1 \subset \cdots \subset F_n \subseteq \mathbb{R}$ and $a_i \in F_i$ for $1 \leq i \leq n$ such that

(i) $K \subseteq F_n$;
(ii) $F_i = F_{i-1}(a_i)$;
(iii) $a_i^2 \in F_{i-1}$.

Remark. By conditions (ii) and (iii), F_i/F_{i-1} is at most a quadratic extension. Then, by the tower law, $F_n/\mathbb{Q}$ has degree a power of two, so $K/\mathbb{Q}$ is a finite extension with degree a power of two.

Theorem. If x is constructible, $\mathbb{Q}(x)$ is constructible.

Proof. Let $K = \mathbb{Q}(x)$. We show that if (x, y) can be constructed with k steps, $\mathbb{Q}(x, y)$ is a constructible extension of $\mathbb{Q}$. By induction, suppose $\mathbb{Q} = F_0 \subset \cdots \subset F_n$ satisfy conditions (ii) and (iii) such that the coordinates of the points obtained after $k - 1$ constructions lie in F_n.

The intersection point of two lines has coordinates given by rational functions of the coordinates of the points P_i, Q_i with rational coefficients. In particular, if the kth construction is of this type, the intersection point has coordinates in F_n. We can similarly see that the intersection points of two circles and the intersection points of a line and a circle have coordinates given by quadratic equations $a \pm b\sqrt{e}, c \pm d\sqrt{e}$, where a, b, c, d, e are rational functions of the coordinates P_i, Q_i. Thus the new points have coordinates which lie in $F_n(\sqrt{e})$, a constructible extension of $\mathbb{Q}$ as required. □

Corollary. If x is constructible, x is algebraic over $\mathbb{Q}$ and the degree of the minimal polynomial is a power of two.

Remark. One can show that if $\mathbb{Q}(x)$ is constructible, we also have x is constructible, so the above theorem is a bi-implication. However, this will not be required for our purposes in this course.

2.7 Classical problems

Theorem. It is impossible to square the circle.

Proof. The statement is to construct a square with area equal to that of a given circle. In particular, we must construct $\sqrt{\pi}$. Suppose such a construction can occur. Then π is also constructible. But π is transcendental and hence inconstructible. □

Theorem. It is impossible to duplicate the cube.

Proof. To duplicate the cube, one must be able to construct $\sqrt[3]{2}$. The minimal polynomial of $\sqrt[3]{2}$ is $X^3 - 2$. This can be easily checked with Eisenstein's criterion. Since the minimal polynomial is of degree not a power of two, $\sqrt[3]{2}$ is inconstructible. □

Theorem. It is impossible to trisect a given angle.

Proof. If we can trisect any constructible angle, we can in particular trisect the (constructible) angle $\frac{2\pi}{3}$, for example to construct a regular nonagon. Then the angle $\frac{2\pi}{9}$ would be constructible, so its sine and cosine would be constructible. By the triple angle formula for cosine,

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \implies 4\cos\left(\frac{2\pi}{9}\right)^3 - 3\cos\left(\frac{2\pi}{9}\right) = \cos\left(\frac{2\pi}{3}\right)$$

Hence $\cos\left(\frac{2\pi}{9}\right)$ is a root of $8X^3 - 6X + 1$. In particular, $2\cos\left(\frac{2\pi}{9}\right) - 2$ is a root of $X^3 + 6X^2 + 9X + 3$, which can be shown to be irreducible by Eisenstein's criterion. But this has degree 3, so $\deg_{\mathbb{Q}} \cos\left(\frac{2\pi}{9}\right) = 3$, so this is inconstructible. In particular, the regular nonagon is inconstructible. □

We will later prove the following theorem.

Theorem (Gauss)**.** A regular n-gon is is constructible if and only if n is the product of a power of two and distinct *Fermat primes*, which are the primes of the form $2^{2^k} + 1$.

3 Types of field extensions

3.1 Fields from polynomials

Suppose K is a field and $f \in K[T]$. We wish to find an extension L/K of degree as small as possible such that f is expressible as a product of linear factors in $L[T]$.

Example. Let $K = \mathbb{Q}$. Then by the fundamental theorem of algebra, a monic polynomial $f \in \mathbb{Q}[T]$ is expressible as a product of n linear factors $(T - x_i)$ in $\mathbb{C}[T]$. One example of such a field extension is $L = \mathbb{Q}(x_1, \dots, x_n)$, which is a finite extension of $\mathbb{Q}$.

We will later give another proof of the fundamental theorem of algebra using techniques from Galois theory.

Example. Let $K = \mathbb{F}_p$, and f is irreducible and has degree $d > 1$. Since there is no ambient field structure, explicitly finding L is more challenging. We will first find an extension in which f has at least one root, and then use induction.

Theorem. Let f be a monic irreducible polynomial. Let $L_f = {}^{K[T]}\!/_{(f)}$. Since f is irreducible, (f) is maximal, hence L_f is a field. Let $t \in L_f$ be the residue class T modulo (f). Then L_f/K is a finite field extension of degree $\deg f$, and f is the minimal polynomial for t.

We have thus constructed a field extension of K for which f has at least a single root. Recall that if x is algebraic over K, then $K(x) \cong {}^{K[T]}\!/_{(f)}$ where f is minimal for x.

Definition. Let K be a field, and $L/K, M/K$ are field extensions. A *K-homomorphism* or *K-embedding* from L to M is a field homomorphism $\sigma : L \to M$ such that $\sigma|_K = \mathrm{id}_K$.

The naming 'K-embedding' is justified because any field homomorphism is injective.

Theorem. Let $f \in K[T]$ be irreducible, and L/K a field extension. Then:

(i) If $x \in L$ is a root of f, there exists a unique K-homomorphism $\sigma : L_f = {}^{K[T]}\!/_{(f)} \to L$ such that $t = T + (f) \mapsto x$.

(ii) Every K-homomorphism $\sigma : L_f \to L$ arises in this way.

Hence, we have a bijection between K-homomorphisms $\sigma : L_f \to L$ and the set of roots of f in L. In particular, there are at most $\deg f$-many K-homomorphisms.

Proof. Let $x \in L$ be a root of f. We define the K-homomorphism $\sigma : {}^{K[T]}\!/_{(f)} \to L$ by $\sigma(T) = x$. Conversely, suppose $\sigma : {}^{K[T]}\!/_{(f)} \to L$ is a K-homomorphism. Then $\sigma(T)$ is a root of f, because $f(\sigma(T)) = \sigma(f(T)) = \sigma(0) = 0$. So the two definitions are inverses, so we have a one-to-one correspondence as required. □

Corollary. Let $L = K(x)$ for some x algebraic over K. Then there exists a unique isomorphism $\sigma : L_f \to K(x)$ such that $\sigma(t) = x$, where f is minimal for x over K.

Definition. Let x, y be algebraic over K. We say x, y are *K-conjugate* if they have the same minimal polynomial over K.

By the corollary above, $K(x)$ and $K(y)$ are isomorphic to L_f where f is minimal for x and y over K.

Corollary. Algebraic elements x, y are K-conjugate if and only if there exists a K-isomorphism $\sigma : K(x) \to K(y)$ such that $\sigma(x) = y$.

Proof. The above corollary shows the forward direction. Conversely, for all $g \in K[T]$, we have $\sigma(g(x)) = g(\sigma(x))$ so they have the same minimal polynomial. □

Informally, the roots of an irreducible polynomial are algebraically indistinguishable.

It can be useful for inductive arguments to have a generalisation of the above theorem.

Definition. Let $L/K, L'/K'$ be field extensions, and let $\sigma : K \to K'$ be a field homomorphism. Let $\tau : L \to L'$ be a field homomorphism such that $\tau(x) = \sigma(x)$ for all $x \in K$. Then we say τ is a *σ-homomorphism* from L to L'. We also say τ *extends* σ, or that σ is the *restriction* of τ to K.

We can now define the following variant of the previous theorem.

Theorem. Let $f \in K[T]$ be irreducible, and $\sigma : K \to L$ be a field homomorphism. Let σf be the polynomial obtained by applying σ to the coefficients of f.

(i) If $x \in L$ is a root of f, there exists a unique σ-homomorphism $\tau : L_f \to L$ such that $\tau(t) = x$.

(ii) Every σ-homomorphism $L_f \to L$ is of this form.

Therefore there is a bijection between the σ-homomorphisms $L_f \to L$ and the roots of f in L.

Example. Let $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$, and $L = \mathbb{C}$. Let $\sigma : K \to L$ be the homomorphism such that $\sigma(x + y\sqrt{2}) = x - y\sqrt{2}$. Then let $f = T^2 - (1 + \sqrt{2})$. Then the map $\tau : L_f \to \mathbb{C}$ must satisfy $\tau(t) = \pm\sqrt{1 - \sqrt{2}} = \pm i\sqrt{\sqrt{2} - 1} \in \mathbb{C}$. If instead we let $\sigma(x + y\sqrt{2}) = x + y\sqrt{2}$, we have $\tau(t) = \pm\sqrt{\sqrt{2} + 1}$, which are both real.

3.2 Splitting fields

Definition. Let $f \in K[T]$ be a nonzero polynomial that is not necessarily irreducible. We say that an extension L/K is a *splitting field* for f over K if

(i) f splits into linear factors in $L[T]$;

(ii) $L = K(x_1, \dots, x_n)$, where the x_i are the roots of f in L.

Remark. The second criterion ensures that f does not split into linear factors in any proper subfield of L. Note that any splitting field is finite, because the adjoined elements are algebraic.

Theorem. Every nonzero polynomial has a splitting field.

Proof. Let $f \in K[T]$. We prove this by induction on the degree of f, but allow K to vary. If f is constant, there is nothing to prove, since K is already a splitting field. Suppose that for all fields K' and all polynomials in $K'[T]$ of degree less than f, there is a splitting field. Consider an irreducible factor g of f, and consider $K' = L_g = {}^{K[T]}\!/_{(g)}$. Let $x_1 = T + (g)$. Then $g(x_1) = 0$, so $f(x_1) = 0$, hence $f = (T - x_1)f_1$, where $f_1 \in K'[T]$. By induction, there exists a splitting field L for f_1 over K' since $\deg f_1 < \deg f$. Let $x_2, \dots, x_n \in L$ be the roots of f_1 in L. Then f splits into linear factors in L with roots $\{x_1, x_2, \dots, x_n\}$. Because L is a splitting field for f_1 over K', we have $L = K'(x_2, \dots, x_n) = K(x_1)(x_2, \dots, x_n) = K(x_1, \dots, x_n)$, so L is a splitting field for f. □

Remark. If $K \subseteq \mathbb{C}$, we already know by the fundamental theorem of algebra that any polynomial over K has a subfield of $\mathbb{C}$ as its splitting field.

Theorem. Let $f \in K[T]$ be a polynomial and L/K be a splitting field for f. Then let $\sigma : K \to M$ be a field homomorphism such that σf splits in $M[T]$. Then
(i) σ can be extended to a homomorphism $\tau : L \to M$;
(ii) if M is a splitting field for σf over σK, then any $\tau : L \to M$ is an isomorphism.
In particular, any two splitting fields are K-isomorphic.

Remark. When constructing the splitting field for a polynomial, we had choice in which irreducible factors to consider first. It is not clear, without this theorem, that two splitting fields have the same degree.

Note that we can have different $\tau_1, \tau_2 : L \to M$ for splitting fields L, M of f over K.

Proof. We will prove (i) by induction on $[L : K]$. If $n = 1$, we have $L = K$ and there is nothing to prove. Suppose $x \in L \setminus K$ is a root of an irreducible factor g of f in K, so $\deg g > 1$. Let $y \in M$ be a root of $\sigma g \in M[T]$, which exists because σf splits in M. Then, there exists $\sigma_1 : K(x) \to M$ such that $\sigma_1(x) = y$, and σ_1 extends σ. Then, $[L : K(x)] < [L : K]$, so by induction, $\sigma_1 : K(x) \to M$ can be extended to $\tau : L \to M$, because L is a splitting field for f over $K(x)$. This τ therefore extends σ as required.

To prove (ii), suppose M is a splitting field for σf over σK. Let τ be as in (i), and $\{x_i\}$ be the roots of f in L. Then the roots of σf in M are $\{\tau(x_i)\}$. Since M is a splitting field, $M = \sigma K(\{\tau(x_i)\}) = \tau L$ as $L = K(\{x_i\})$. So τ is an isomorphism.

If $K \subseteq M$ and σ is the inclusion homomorphism, τ is a K-isomorphism. □

Example. Let $f = T^3 - 2 \in \mathbb{Q}[T]$. This has splitting field $L = \mathbb{Q}(\sqrt[3]{2}, \omega) \subseteq \mathbb{C}$ where $\omega = e^{\frac{2\pi i}{3}}$. We know $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$, but $\omega \notin \mathbb{R}$ and $\omega^2 + \omega + 1 = 0$, so $[L : \mathbb{Q}(\sqrt[3]{3})] = 2$ giving $[L : \mathbb{Q}] = 6$ by the tower law. In particular, adjoining a single root to $\mathbb{Q}$ is not enough to generate L.

Example. Let $f = \frac{T^5-1}{T-1} = T^4 + \cdots + T + 1 \in \mathbb{Q}[T]$. Let $z = e^{\frac{2\pi i}{5}}$, then this is the minimal polynomial of z. We find $f = \prod_{1 \le a \le 4}(T - z^a)$, so $\mathbb{Q}(z)$ is already a splitting field for f over $\mathbb{Q}$, and $[\mathbb{Q}(z) : \mathbb{Q}] = 4$.

Example. Let $f = T^3 - 2 \in \mathbb{F}_7[T]$. This is irreducible because 2 is not a cube in $\mathbb{F}_7$. Consider $L = {}^{\mathbb{F}_7[X]}\!/_{X^3-2} = \mathbb{F}_7(x)$, so $x^3 = 2$. Since $2^3 = 4^3 = 1$ in $\mathbb{F}_7$, we have $(2x)^3 = (4x)^3 = 2$, so $x, 2x, 4x$ are roots of f in L. In particular, L is a splitting field for f, since $f = (T-x)(T-2x)(T-4x)$; here, adjoining one root is enough to make f split.

3.3 Normal extensions

Definition. An extension L/K is a *normal extension* if it is algebraic and for all $x \in L$, the minimal polynomial splits in L.

Remark. This condition is equivalent to the statement that for every $x \in L$, L contains a splitting field for x. In other words, if an irreducible polynomial $f \in K[T]$ has a single root in L, it splits and has all roots in L.

Theorem. Let L/K be a finite extension. Then L is normal over K if and only if L is a splitting field for some (not necessarily irreducible) polynomial $f \in K[T]$.

Proof. Suppose L is normal. Then $L = K(x_1, \dots, x_n)$ since L is algebraic. Then the minimal polynomial $m_{x_i,K}$ of each x_i over K splits in L. L is generated by the roots of $\prod_i m_{x_i,K}$, so L is a splitting field for f.

For the converse, suppose L is a splitting field for $f \in K[T]$. Let $x \in L$, and let $g = m_{x,K}$ be its minimal polynomial. We want to show that g splits in L. Let M be a splitting field for g over L, and let $y \in M$ be a root of g. We want to show $y \in L$.

Since L is a splitting field for f over K, L is a splitting field for f over $K(x)$, and $L(y)$ is a splitting field for f over $K(y)$. Now, there exists a K-isomorphism between $K(x)$ and $K(y)$, because x, y are roots of the same irreducible polynomial g. By the uniqueness of splitting fields, $[L : K(x)] = [L(y) : K(y)]$. Multiplying by $[K(x) : K]$, we find $[L : K] = [L(y) : K]$ because $[K(y) : K] = [K(x) : K]$ as they are roots of the same irreducible polynomial. Hence $[L(y) : L] = 1$, so $y \in L$ as required. □

Corollary (normal closure)**.** Let L/K be a finite extension. Then there exists a finite extension M/L such that M/K is normal, and if $L \subseteq M' \subseteq M$ and M'/K is normal, $M = M'$. Moreover, any two such extensions M are L-isomorphic.

Such an M is said to be a *normal closure* of L/K.

Proof. Let $L = K(x_1, \dots, x_k)$, and $f = \prod_{i=1}^{k} m_{x_i,K} \in K[T]$. Then let M be a splitting field of f over L. Then, since the x_i are roots of f, M is also a splitting field for f over K. So M/K is normal.

Let M' be such that $L \subseteq M' \subseteq M$ and M'/K be normal. Then as $x_i \in M'$, the minimal polynomial $m_{x_i,K}$ splits in M'. So $M' = M$.

Any normal extension M/K must contain a splitting field for f, and by the minimality condition, M must be a splitting field. By uniqueness of splitting fields, any two such extensions are L-isomorphic as required. □

3.4 Separable polynomials

Recall that over $\mathbb{C}$, a root x of a polynomial is said to be a multiple zero when its derivative vanishes at x. Over arbitrary fields, the same is true, but the analytic concept of derivative must be replaced with an algebraic process.

Definition. The *formal derivative* of a polynomial $f(T) = \sum_{i=0}^{d} a_i X^i$ is

$$f'(T) = \sum_{i=1}^{d} i a_i X^{i-1}$$

Remark. One can check from the definition that the familiar rules $(f+g)' = f'+g'$, $(fg)' = f'g+fg'$, and $(f^n)' = nf'f^{n-1}$ hold.

Example. Consider a field K of characteristic $p > 0$, and let $f = T^p + a_0$. Then $f' = 0$, so a non-constant polynomial can have a zero derivative.

Proposition. Let $f \in K[T]$, L/K be a field extension, and $x \in L$ a root of f. Then x is a simple root if and only if $f'(x) \neq 0$.

Proof. We can write $f = (T-x)g \in L[T]$. Then $f' = g + (T-x)g'$, so $f'(x) = g(x)$. In particular, $f'(x) \neq 0$ if and only if $(T-x)$ does not divide g, which is the criterion that x is a simple root of f. □

Definition. A polynomial $f \in K[T]$ is *separable* if it splits into distinct linear factors in a splitting field. Equivalently, it has $\deg f$ distinct roots.

Corollary. f is separable if and only if the greatest common divisor of f and f' is 1.

For convenience, we will take $\gcd(f,g)$ to be the unique monic polynomial h such that $(h) = (f,g)$. Then since $K[T]$ is a Euclidean domain, we can compute a representation $h = af+bg$ for polynomials a, b. Note that $\gcd(f,g)$ is invariant under a field extension, because Euclid's algorithm does not depend on the ambient field structure.

Proof. We can replace K by a splitting field of f, so we can factorise f into a product of linear factors in K. The two are separable if f, f' have no common root, which is true if and only if $\gcd(f,f') = 1$. □

Example. Let K have characteristic $p > 0$, and let $f = T^p - b$ for $b \in K$. Then $f' = 0$, so $\gcd(f,f') = f \neq 1$. Hence f is inseparable. Let L be an extension of K containing a pth root $a \in L$ of b, so $a^p = b$. Then $f = (T-a)^p = T^p + (-a)^p = T^p - b$. In particular, f has only one root in a splitting field.

If b is not a pth power in K, then f is irreducible. This is seen on the example sheets.

Theorem. Let $f \in K[T]$ be an irreducible polynomial. Then f is separable if and only if $f' \neq 0$.
In addition, if K has characteristic zero, every irreducible polynomial $f \in K[T]$ is separable. If K has positive characteristic $p > 0$, an irreducible polynomial $f \in K[T]$ is inseparable if and only if $f(T) = g(T^p)$ for some $g \in K[T]$.

Proof. Without loss of generality, we can assume f is monic. Then, since f is irreducible, the greatest common divisor $\gcd(f, f')$ is either f or 1. If $\gcd(f, f') = f$, then $f' = 0$ by considering the degree.

For a polynomial f, we can write $f = \sum_{i=0}^{d} a_i T^i$ and $f' = \sum_{i=1}^{d} i a_i T^{i-1}$, so $f' = 0$ if and only if $ia_i = 0$ for all $1 \leq i \leq d$. In particular, if K has characteristic zero, this is true if and only if $a_i = 0$ for all $1 \leq i \leq d$, so $f = a_0$ is a constant so not irreducible. If K has characteristic $p > 0$, the requirement is that $a_i = 0$ for all i not divisible by p, or equivalently, $f(T) = g(T^p)$. □

3.5 Separable extensions

Definition. Let L/K be a field extension. We say $x \in L$ is *separable* over K if x is algebraic and its minimal polynomial f is separable over K. L is *separable* over K if all elements x are separable over K.

Theorem. Let x be algebraic over K, and L/K be an extension in which the minimal polynomial $m_{x,K}$ splits. Then x is separable over K if and only if there are exactly $\deg x$ K-homomorphisms from $K(x)$ to L.

Proof. The number of K-homomorphisms from $K(x)$ to L is the number of roots of $m_{x,K}$ in L. This is equal to the degree of x if and only if x is separable. □

Let $\mathrm{Hom}_K(L, M)$ be the set of K-homomorphisms from L to M. Note that not all K-linear maps from L to M are K-homomorphisms.

Theorem (counting embeddings)**.** Let $L = K(x_1, \dots, x_k)$ be a finite extension of K, so the x_i are algebraic. Let M/K be any field extension. Then $|\mathrm{Hom}_K(L, M)| \leq [L : K]$, with equality if and only if
(i) for all i, the minimal polynomial $m_{x_i,K}$ splits into linear factors in M; and
(ii) all the x_i are separable over K.

Remark. The conditions (i) and (ii) are equivalent to the statement that $m_{x_i,K}$ split into distinct linear factors over M. There is a variant of this theorem: let $\sigma : K \to M$ be a field homomorphism, then $|\mathrm{Hom}_\sigma(L, M)| \leq |L : K|$, and equality holds if and only if the $\sigma m_{x_i,K}$ split into distinct linear factors over M.

Proof. We prove this by induction on k. The case $k = 0$ is trivial. Let $K_1 = K(x_1)$ and write $d = \deg_K x_1 = [K_1 : K]$. Then the number of K-homomorphisms from K_1 to M, denoted $e = |\mathrm{Hom}_K(K_1, M)|$, is the number of roots of $m_{x_1,K}$ in M. Let $\sigma : K_1 \to M$ be a K-homomorphism. By

the inductive hypothesis, there exist at most $[L : K_1]$ extensions of σ to a K-homomorphism $L \to M$. Hence the number of K-homomorphisms from L to M is at most $e[L : K_1] \leq d[L : K_1] = [L : K]$.

If equality holds, then $e = d$, and so $m_{x_1,K}$ splits into d distinct linear factors in M, so (i) and (ii) hold for x_1. Replacing x_1 with an arbitrary x_i, one implication follows. Conversely, suppose conditions (i) and (ii) hold. Then, by the previous theorem, there are d distinct homomorphisms from K_1 to M. Conditions (i) and (ii) still hold over K_1, then by induction, each $\sigma : K_1 \to M$ has $[L : K_1]$ extensions to a homomorphism $L \to M$. Hence $|\mathrm{Hom}_K(L, M)| = [L : K]$ as required. □

Theorem (separably generated implies separable)**.** Let $L = K(x_1, \dots, x_k)$ be a finite extension of K. Then L/K is a separable extension if and only if each x_i is separable over K.

Proof. If L/K is separable, the x_i are separable by definition. Suppose the x_i are separable. Let M be a normal closure of L/K, so the splitting field of the product of the $m_{x_i,K}$ over L. By the counting embeddings theorem, conditions (i) and (ii) are satisfied so $|\mathrm{Hom}_K(L, M)| = [L : K]$. But if $x \in L$, $L = K(x, x_1, \dots, x_k)$, so x is separable. □

Corollary. Let $x, y \in L$, and L/K a field extension. If x, y are separable over K, so are $x + y, xy, x^{-1}$ for $x \neq 0$.

Proof. Consider the fields $K(x, y)$ and $K(x)$. These are separable extensions of K. In particular, $\{x \in L \mid x \text{ separable over } K\}$ is a subfield of L. □

Theorem (primitive element theorem for separable extensions)**.** Let K be an infinite field and $L = K(x_1, \dots, x_k)$ be a finite separable extension. Then there exists $x \in L$ such that $L = K(x)$. In particular, x is separable over K.

Proof. It suffices to consider the case when $k = 2$, because if we can turn $K(x, y)$ into $K(z)$ for $z \in K(x, y)$, we can perform this inductively. Let $L = K(x, y)$ with x, y separable over K. Let $n = [L : K]$, and let M be a normal closure for L/K. Then there exist n distinct K-homomorphisms $\sigma_i : L \to M$. Let $a \in K$, and consider $z = x + ay$. We will choose a such that $L = K(z)$.

Since $L = K(x, y)$, we have $\sigma_i(x) = \sigma_j(x)$ and $\sigma_i(y) = \sigma_j(y)$ implies $i = j$. Consider $\sigma_i(z) = \sigma_i(x) + a\sigma_i(y)$. If $\sigma_i(z) = \sigma_j(z)$, we must have $\big(\sigma_i(x) - \sigma_j(x)\big) + a\big(\sigma_i(y) - \sigma_j(y)\big) = 0$. If $i \neq j$, at least one of the parenthesised terms is nonzero. Therefore there is at most one $a \in K$ such that $\sigma_i(z) = \sigma_j(z)$. Since K is infinite, there exists $a \in K$ such that all of the $\sigma_i(z)$ are distinct. But then $\deg_K z = n$, so $L = K(z)$. □

Theorem. Let L/K be an extension of finite fields. Then $L = K(x)$ for some $x \in L$.

Proof. The multiplicative group $L^\times$ is cyclic. Let x be a generator of this group. Then $L = K(x)$, since every nonzero element is a power of x. □

4 Galois theory

4.1 Field automorphisms

Definition. A bijective homomorphism from a field to itself is called an *automorphism*. The set of automorphisms of a field L forms a group $\mathrm{Aut}(L)$ under composition: $(\sigma\tau)(x) = \sigma(\tau(x))$. This is called the *automorphism group of* L. Let $S \subseteq \mathrm{Aut}(L)$. Then, we define

$$L^S = \{x \in L \mid \forall \sigma \in S, \sigma(x) = x\}$$

This is a subfield of L, known as the *fixed field of* S, since each σ is a homomorphism.

Example. Let $L = \mathbb{C}$ and σ be the complex conjugation automorphism. Then the fixed field of $\{\sigma\}$ is $\mathbb{C}^{\{\sigma\}} = \mathbb{R}$.

Definition. Let L/K be a field extension. We define $\mathrm{Aut}(L/K)$ to be the set of K-automorphisms of L, so $\mathrm{Aut}(L/K) = \{\sigma \in \mathrm{Aut}(L) \mid \forall x \in K, \sigma(x) = x\}$. Equivalently, $\sigma \in \mathrm{Aut}(L)$ is an element of $\mathrm{Aut}(L/K)$ if $K \subseteq L^{\{\sigma\}}$. $\mathrm{Aut}(L/K)$ is a subgroup of $\mathrm{Aut}(L)$.

Theorem. Let L/K be a finite extension. Then $|\mathrm{Aut}(L/K)| \leq [L : K]$.

Proof. Let $M = L$, then $\mathrm{Hom}_K(L, M) = \mathrm{Aut}(L/K)$, which has at most $[L : K]$ elements. □

Proposition. If $K = \mathbb{Q}$ or $K = \mathbb{F}_q$, $\mathrm{Aut}(K) = \{1\}$.

Proof. $\sigma(1_K) = 1_K$ hence $\sigma(n_K) = n_K$. □

In particular, $\mathrm{Aut}(L) = \mathrm{Aut}(L/K)$ where K is the prime subfield of L.

4.2 Galois extensions

We need to define a notion of when an extension L/K has 'many symmetries'.

Definition. An extension L/K is a *Galois extension* if it is algebraic, and $L^{\mathrm{Aut}(L/K)} = K$.

Remark. If $x \in L \setminus K$, there is a K-automorphism $\sigma : L \to L$ such that $x \neq \sigma(x)$.

Example. $\mathbb{C}/\mathbb{R}$ is a Galois extension, since the fixed field of complex conjugation is $\mathbb{R}$. Similarly, $\mathbb{Q}(i)/\mathbb{Q}$ is a Galois extension.

Example. Let $K/\mathbb{F}_p$ be a finite extension, so K is a finite field. The Frobenius automorphism of K, given by $\varphi_p(x) = x^p$, has fixed field

$$K^{\{\varphi_p\}} = \{x \in K \mid x \text{ a root of } T^p - T\}$$

But since this has at most p roots, and each element of $\mathbb{F}_p$ is a root, the fixed field is exactly $\mathbb{F}_p$. So $K^{\mathrm{Aut}(K/\mathbb{F}_p)} = \mathbb{F}_p$, so this is a Galois extension.

Definition. Let L/K be a Galois extension. We write $\mathrm{Gal}(L/K)$ for the automorphism group $\mathrm{Aut}(L/K)$, called the *Galois group of* L/K.

Theorem (classification of finite Galois extensions)**.** Let L/K be a finite extension, and let $G = \mathrm{Aut}(L/K)$, then the following are equivalent.

(i) L/K is a Galois extension, so $K = L^G$.
(ii) L/K is normal and separable.
(iii) L is a splitting field of a separable polynomial in K.
(iv) $|\mathrm{Aut}(L/K)| = [L : K]$.

If this holds, the minimal polynomial of any $x \in L$ over K is $m_{x,K} = \prod_{i=1}^{r}(T - x_i)$, where $\{x_1, \dots, x_r\}$ is the orbit of G on x.

Proof. (i) implies (ii) and the minimal polynomial result. Let $x \in L$, and $\{x_1, \dots, x_r\}$ be the orbit of G on x. Let $f = \prod_{i=1}^{r}(T - x_i)$. Then $f(x) = 0$. Since G permutes the x_i, the coefficients of f are fixed by G. By assumption, the coefficients of f lie in K, so the minimal polynomial of x must divide f. Since $m_{x,K}(\sigma(x)) = \sigma(m_{x,K}(x)) = 0$, so every x_i is a root of the minimal polynomial of $m_{x,K}$. So f is exactly the minimal polynomial as required. $m_{x,K}$ is a separable polynomial and splits in L. So L/K is normal and separable.

(ii) implies (iii). Since splitting fields are normal extensions, L is a splitting field for some polynomial $f \in K[T]$. Write $f = \prod_{i=1}^{r} q_i^{e_i}$ where the q_i are distinct irreducible polynomials, and $e_i \geq 1$. Since L and K are separable, the q_i are separable as they are irreducible, so $g = \prod_{i=1}^{r} q_i$ is separable and L is also a splitting field for g.

(iii) implies (iv). Let $L = K(x_1, \dots, x_k)$ be the splitting field of a separable polynomial $f \in K[T]$ with roots x_i. By the theorem on counting embeddings with $M = L$, since $m_{x_i,K} \mid f$, conditions (i) and (ii) in the theorem are satisfied, and we find $|\mathrm{Aut}(L/K)| = |\mathrm{Hom}_K(L, M)| = [L : K]$.

(iv) implies (i). Suppose $|\mathrm{Aut}(L/K)| = |G| = [L : K]$. Note that $G \subseteq \mathrm{Aut}(L/L^G) \subseteq \mathrm{Aut}(L/K)$, so these inclusions are both equalities. So $G = \mathrm{Aut}(L/L^G)$, so $[L : K] = |G| \leq [L : L^G]$. But since $L^G \supseteq K$, we must have equality by the tower law. □

Corollary. Let L/K be a finite Galois extension. Then $L = K(x)$ for some $x \in L$ which is separable over K, and has degree $[L : K]$.

Proof. By (ii) above, L/K is separable. Then the primitive element theorem implies that $L = K(x)$ for some x. □

4.3 Galois correspondence

Theorem (Galois correspondence: part (a))**.** Let L/K be a finite Galois extension with $G = \mathrm{Gal}(L/K)$. Suppose F is another field, and $K \subseteq F \subseteq L$. Then L/F is also a Galois extension where $\mathrm{Gal}(L/F) \leq \mathrm{Gal}(L/K)$. The map $F \mapsto \mathrm{Gal}(L/F)$ is a bijection between the set of intermediate fields F and the set of subgroups of $H \leq \mathrm{Gal}(L/K)$. The inverse of this map is $H \mapsto L^H$. This bijection reverses inclusions, and if $F = L^H$, we have $[F : K] = (G : H)$.

Proof. Let $x \in L$. Then $m_{x,F} \mid m_{x,K}$ in $F[T]$. As $m_{x,K}$ splits into distinct linear factors in L so does $m_{x,F}$. So L/F is normal and separable, and hence a Galois extension as required. By definition, $\mathrm{Gal}(L/F) \leq \mathrm{Gal}(L/K)$.

To check the map $F \mapsto \mathrm{Gal}(L/F)$ is a bijection with the given inverse, we first consider a field F, and its image $L^{\mathrm{Gal}(L/F)}$ under both maps. We have $L^{\mathrm{Gal}(L/F)} = F$, since L/F is Galois as required. Conversely, suppose $H \leq \mathrm{Gal}(L/F)$, and consider its image $\mathrm{Gal}(L/L^H)$. To show $\mathrm{Gal}(L/L^H) = H$, it suffices to show that $[L : L^H] \leq |H|$, because certainly $H \leq \mathrm{Gal}(L/L^H)$ and $|\mathrm{Gal}(L/L^H)| \leq [L : L^H]$. By the previous corollary, $L = L^H(x)$ for some x, and $f = \prod_{\sigma \in H}(T - \sigma(x)) \in L^H[T]$ is a polynomial with x as a root. In particular, $[L : L^H] = \deg_{L^H}(x) \leq \deg f = |H|$. So we have a bijection as claimed.

Suppose $F \subseteq F'$ are fields between K and L. Then $\mathrm{Gal}(L/F') \subseteq \mathrm{Gal}(L/F)$, so the bijection reverses inclusions. Finally, if $F = L^H$, we have $[F : K] = \frac{[L:K]}{[L:F]} = \frac{|\mathrm{Gal}(L/K)|}{|\mathrm{Gal}(L/F)|} = \frac{|G|}{|H|} = (G : H)$. □

Theorem (Galois correspondence: part (b))**.** Let $H \leq G$ be a subgroup of a Galois group $G = \mathrm{Gal}(L/K)$. Then $\sigma H \sigma^{-1}$ corresponds to the field σL^H.

Proof. Under the Galois correspondence, $\sigma H \sigma^{-1}$ corresponds to its fixed field

$$L^{\sigma H \sigma^{-1}} = \left\{x \in L \mid \sigma\tau\sigma^{-1}(x) = x \text{ for all } \tau \in H\right\}$$

Note that $\sigma\tau\sigma^{-1}(x) = x$ if and only if $\tau\sigma^{-1}(x) = \sigma^{-1}(x)$, so $\tau(y) = y$ for $x = \sigma(y)$. Hence $x \in L^{\sigma H \sigma^{-1}}$ if and only if there exists $y \in L^H$, $x = \sigma(y)$. Therefore $L^{\sigma H \sigma^{-1}} = \sigma L^H$ as required. □

Theorem (Galois correspondence: part (c))**.** Let $H \leq G = \mathrm{Gal}(L/K)$. Then the following are equivalent.

(i) L^H/K is Galois;
(ii) L^H/K is normal;
(iii) for all $\sigma \in G$, $\sigma L^H = L^H$;
(iv) H is a normal subgroup of G.

If so, $\mathrm{Gal}(L^H/K) = {}^{\mathrm{Gal}(L/K)}\!/_{H} = {}^{G}\!/_{H}$.

Proof. (i) and (ii) are equivalent. L/K is separable since it is Galois. So L^H/K is also separable.

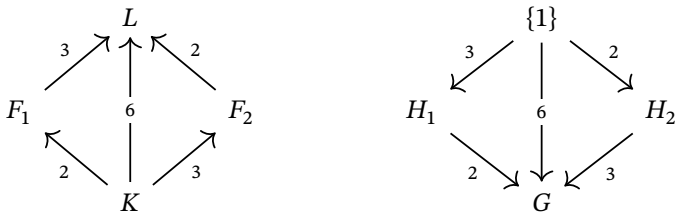
(iii) and (iv) are equivalent. Let $F = L^H$, and let $x \in F$. Then the set of roots of $m_{x,K}$ is the orbit of x under G, so the minimal polynomial splits in F if and only if for all $\sigma \in G$, $\sigma(x) \in F$. As this holds for all $x \in F$, F is normal if and only if $\sigma F \subseteq F$. Since $[\sigma F : K] = [F : K]$, as F and σF are K-isomorphic,

this holds if and only if $\sigma F = F$. By part (b) of the Galois correspondence, this is equivalent to the statement that $\sigma H \sigma^{-1} = H$ for all σ, so H is normal.

If any of the above hold, for all $\sigma \in G$, we have $\sigma F = F$, so we have homomorphisms $G \to \mathrm{Gal}(F/K)$ given by the restriction of $\sigma \in G$ to F. Its kernel is H. Then from the isomorphism theorem, G/H is isomorphic to a subgroup of $\mathrm{Gal}(F/K)$. This must be an isomorphism because $[F : K] = (G : H)$. □

Example. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = e^{\frac{2\pi i}{3}}$. L is a splitting field for $T^3 - 2$ with $[L : \mathbb{Q}] = 6$. Since $T^3 - 2$ is a separable polynomial, L is the splitting field of a separable polynomial and hence Galois. Therefore $G = \mathrm{Gal}(L/\mathbb{Q})$ has order 6.

We have the subfields $F_1 = \mathbb{Q}(\omega), F_2 = \mathbb{Q}(\sqrt[3]{2})$, where $[F_1 : \mathbb{Q}] = 2$ and $[F_2 : \mathbb{Q}] = 3$. In the following diagram, the arrows on the left hand side are annotated with the degrees of an extensions, and the arrows on the right hand side are labelled with the index of the relevant subgroup.



By the classification of finite groups of order 6, G is isomorphic either to C_6 or S_3. $F_2 = \mathbb{Q}(\sqrt[3]{2})$ is not a normal extension of $\mathbb{Q}$, because $\omega\sqrt[3]{2} \notin F_2$. So H_2 is not a normal subgroup of G. Since all subgroups of abelian groups are normal, G is not abelian. So $G \cong S_3$. Hence $H_1 \cong A_3$, and H_2 is a transposition, but since all subgroups generated by transpositions are conjugate, we can set $H_2 = \langle (1\ 2) \rangle$.

The other two subgroups are conjugate to H_2, corresponding to the subfields σF_2 where $\sigma \in G$. Hence, these subfields are exactly $\mathbb{Q}(\omega\sqrt[3]{2})$ and $\mathbb{Q}(\omega^2\sqrt[3]{2})$, since the conjugates of $\sqrt[3]{2}$ are exactly the roots of the minimal polynomial. Note that since these are the only subgroups, we have found all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[3]{2}, \omega)$.

There is an easier way to prove $G \cong S_3$. Consider a separable polynomial $f \in K[T]$, and its roots $x_1, \ldots, x_n$ in a splitting field L. Then $G = \mathrm{Gal}(L/K)$ permutes the $\{x_i\}$, because $f(\sigma x_i) = \sigma f(x_i) = 0$. If $\sigma(x_i) = x_i$ for all i, since $L = K(x_1, \ldots, x_n)$, σ must be the identity map. This gives an injective homomorphism from G into S_n. So G is isomorphic to a subgroup of S_n. In our example above, $|G| = 6$ and G is isomorphic to a subgroup of S_3, so $G \cong S_3$.

4.4 Galois groups of polynomials

Definition. Let $f \in K[T]$, and let L be a splitting field for f. There is an action of $\mathrm{Gal}(L/K)$ on the set of roots of f in L. If f has n roots, this action induces a subgroup of permutations of roots $\mathrm{Gal}(f/K) \leq S_n$, called the *Galois group of f over K*.

Remark. $\mathrm{Gal}(f/K) \simeq \mathrm{Gal}(L/K)$ as L is a splitting field for f over K. In particular, $[L : K] = |\mathrm{Gal}(L/K)| = |\mathrm{Gal}(f/K)| \mid n!$.

There exist several methods for finding the Galois group for a particular polynomial.

Proposition. $f \in K[T]$ is irreducible if and only if $\mathrm{Gal}(f/K)$ is *transitive*, so for all $i, j \in \{1, \dots, n\}$, there exists $\sigma \in \mathrm{Gal}(f/K)$ such that $\sigma(i) = j$.

Remark. A subgroup of S_n is transitive if and only if there is exactly one orbit.

Proof. Let x be a root of f in a splitting field L. Then its orbit under $G = \mathrm{Gal}(f/K)$ is exactly the set of roots of $m_{x,K}$. Since $m_{x,K}$ is an irreducible factor of f, $m_{x,K} = f$ if and only if f is irreducible. Conversely, $m_{x,K} = f$ if and only if each root of f is in the orbit of x, which is exactly the statement that G acts transitively on the roots of f. □

Remark. If $G \subseteq S_n$ is transitive, by the orbit-stabiliser theorem, $n \mid |G|$.

Recall that for a monic polynomial $f = \prod_{i=1}^{n}(T - x_i)$, the *discriminant* of f is $\mathrm{Disc}(f) = \Delta^2 \in K$, where $\Delta = \prod_{i<j}(x_i - x_j)$. The discriminant is nonzero if and only if f is separable.

Proposition. Let $\mathrm{char}\, K \neq 2$, and let $f \in K[T]$ be a monic polynomial with splitting field L. Let $G = \mathrm{Gal}(f/K)$. Then the fixed field of $G \cap A_n$ is $K(\Delta)$, where Δ^2 is the discriminant. In particular, $\mathrm{Gal}(f/K) \subseteq A_n$ if and only if the discriminant $\mathrm{Disc}(f)$ is a square.

Proof. Let $\pi \in S_n$. The sign of the permutation is given by

$$\prod_{i<j}(T_{\pi(i)} - T_{\pi(j)}) = \operatorname{sgn}\pi \prod_{i<j}(T_i - T_j)$$

Hence, if $\sigma \in G$, we have $\sigma(\Delta) = \operatorname{sgn}\sigma \cdot \Delta$. Because the characteristic is not 2, $-1 \neq 1$. Since $\Delta \neq 0$, this implies $\Delta \in K$ if and only if $G \subseteq A_n$, and Δ lies in the fixed field F of $G \cap A_n$. Because $[F : K] = (G : G \cap A_n) \in \{1, 2\}$, $F = K(\Delta)$ exactly. □

Example. Let $n = 3$, $f = T^3 + aT + b = \prod_{i=1}^{3}(T - x_i)$ where x_i lie in a splitting field for f. Since there is no T^2 term, $x_3 = -x_1 - x_2$. Hence, $a = x_1x_2 - (x_1 + x_2)^2$, and $b = x_1x_2(x_1 + x_2)$. Therefore,

$$\mathrm{Disc}(f) = [(x_1 - x_2)(2x_1 + x_2)(x_1 + 2x_2)]^2 = -4a^3 - 27b^2$$

In particular, $\mathrm{Gal}(f/K) \subseteq A_3$ if and only if $-4a^3 - 27b^2$ is a square in K.

For example, consider $f = T^3 - 21T - 7 \in \mathbb{Q}[T]$. This is irreducible by Eisenstein's criterion. Its discriminant is $4 \cdot 21^3 - 27 \cdot 7^2 = (27 \cdot 7)^2$, which is a square. So $\mathrm{Gal}(f/K) \subseteq A_3$. Since f is irreducible, $\mathrm{Gal}(f/K)$ is transitive, so its order is divisible by 3. So $\mathrm{Gal}(f/K)$ must be exactly A_3.

Remark. This technique can be used to calculate the Galois group of any cubic polynomial for characteristic not 2, 3, for example.

5 Finite fields

5.1 Construction of finite fields

Every finite field has characteristic $p > 0$, and so it can be regarded as a field extension of $\mathbb{F}_p$. We will classify every finite field and study their Galois theory. Recall that, for a finite field F of characteristic p,

(i) $|F| = p^n$, where $[F : \mathbb{F}_p] = n$;

(ii) $F^\times$ is cyclic, of order $p^n - 1$;

(iii) The Frobenius automorphism $\varphi_p : F \to F$ given by $x \mapsto x^p$ is an automorphism of F.

Theorem. Let p be a prime, and $n \geq 1$. Then there is a finite field with $q = p^n$ elements. Any such field is a splitting field of the polynomial $f = T^q - T$ over $\mathbb{F}_p$. Since splitting fields are unique up to $\mathbb{F}_p$-isomorphism, any two finite fields of the same order are isomorphic.

Proof. Let F be a field with $q = p^n$ elements. Then if $x \in F^\times$, $x^{q-1} = 1$. Hence, for all $x \in F$, $x^q = x$. In particular, f has q distinct roots in F, which are all of the elements of F. So f splits into linear factors in F, and not in any proper subfield, so F is indeed a splitting field for f as required.

Now, we wish to explicitly construct such a field. Let L be a splitting field for $f = T^q - T$ over $\mathbb{F}_p$. Let $F \subseteq L$ be the fixed field of φ_p^n, the map $x \mapsto x^q$. So F is the set of roots of f in L. So $|F| = q$. Therefore, $L = F$ because F has q elements, using the above argument. □

Now that we have shown isomorphism, we simply write $\mathbb{F}_q$ for any finite field of q elements. There is no canonical finite field of a given order in general.

5.2 Galois theory of finite fields

Theorem. The extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois, and the Galois group is cyclic of order n, generated by the Frobenius automorphism φ_p.

Proof. Since $\mathbb{F}_{p^n}$ is the splitting field of the separable polynomial $T^{p^n} - T$, the extension is Galois. Let $G \leq \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ be the subgroup generated by φ_p. Then $\mathbb{F}_{p^n}^G = \{x \mid x^p = x\} = \mathbb{F}_p$, so by the Galois correspondence, G must be the entire group $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. □

Theorem. $\mathbb{F}_{p^n}$ has a unique subfield of order p^m for all $m \mid n$, and no others. If $m \mid n$, then $\mathbb{F}_{p^m} \subseteq \mathbb{F}_{p^n}$ is the fixed field of φ_p^m.

Proof. By the Galois correspondence, it suffices to check the subgroups of $\mathbb{Z}/n\mathbb{Z}$. The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are $m\mathbb{Z}/n\mathbb{Z}$ for $m \mid n$. Hence, the subfields of $\mathbb{F}_{p^m}$ are the fixed fields of the subgroups $\langle \varphi_p^m \rangle$, which have degree equal to the indices $\left(\mathbb{Z}/n\mathbb{Z} : m\mathbb{Z}/n\mathbb{Z}\right) = m$. □

Remark. If $m \mid n$, $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m}) = \langle \varphi_p^m \rangle$, which has order $\frac{n}{m}$.

Theorem. Let $f \in \mathbb{F}_p[T]$ be separable, and let $n = \deg f$. Suppose the irreducible factors of f have degrees $n_1, \ldots, n_r$, so $\sum_{i=1}^r n_i = n$. Then $\mathrm{Gal}(f/\mathbb{F}_p) \subseteq S_n$ is cyclic and generated by an element of cycle type $(n_1, \ldots, n_r)$. In particular, $\left|\mathrm{Gal}(f/\mathbb{F}_p)\right|$ is the least common multiple of the n_i.

Recall that $\pi \in S_n$ has cycle type $(n_1, \dots, n_r)$ if it is a product of r disjoint cycles π_i, each with length n_i.

Proof. Let L be a splitting field for f over $\mathbb{F}_p$. Consider $x_1, \dots, x_n \in L$. Then $\mathrm{Gal}(L/\mathbb{F}_p)$ is cyclic and generated by φ_p. As the irreducible factors g_i of f are the minimal polynomials of the x_i, and the roots of the minimal polynomial of x_i are precisely the orbit of φ_p on x_i, the cycle type must be as required. The order of any such permutation is the lowest common multiple of the lengths of the cycles. □

5.3 Reduction modulo a prime

Theorem. Let $f \in \mathbb{Z}[T]$ be a monic separable polynomial with $\deg f = n$, and let p be a prime. Suppose that the *reduction* $\overline{f} \in \mathbb{F}_p[T]$ of f is also separable. Then $\mathrm{Gal}(\overline{f}/\mathbb{F}_p) \leq \mathrm{Gal}(f/\mathbb{Q})$ as subgroups of S_n.

Remark. The identification of $\mathrm{Gal}(f/\mathbb{Q})$ with a subgroup of S_n depends on the choice of ordering of the roots of f. Choosing a different ordering corresponds to conjugation of $\mathrm{Gal}(f/\mathbb{Q})$ in S_n. The meaning of the statement $\mathrm{Gal}(\overline{f}/\mathbb{F}_p) \leq \mathrm{Gal}(f/\mathbb{Q})$ therefore means that $\mathrm{Gal}(\overline{f}/\mathbb{F}_p)$ is conjugate to a subgroup of $\mathrm{Gal}(f/\mathbb{Q})$ in S_n, not that it is exactly a subgroup.

The following proof is based in algebraic number theory; alternatives are available. The proof is not examinable.

Proof. Let $L = \mathbb{Q}(x_1, \dots, x_n)$ be a splitting field for f, where the x_i are the roots of f. Let $N = [L : \mathbb{Q}]$. Consider $R = \mathbb{Z}[x_1, \dots, x_n]$. Since $f(x_i) = 0$ and f is monic, every element of R is a $\mathbb{Z}$-linear combination of $x_1^{a_1}, \dots, x_n^{a_n}$ where the $a_i < n$ by using f to reduce the degrees. So R is finitely-generated as a $\mathbb{Z}$-module, or equivalently, as an abelian group. R is contained inside $L \simeq \mathbb{Q}^N$. R is torsion-free, so $R \simeq \mathbb{Z}^M$ with $M \leq N$ (in fact, $M = N$).

Then $\overline{R} = R/pR$ has p^M elements. Let $\overline{P}$ be a maximal ideal for $\overline{R}$, which corresponds to an ideal P of R that contains pR. Then $F = R/P \simeq \overline{R}/\overline{P}$ (by the isomorphism theorem) is a finite field with p^d elements for some d. Since R is generated by $x_1, \dots, x_n$, F is generated by $\overline{x}_1, \dots, \overline{x}_n$, where $\overline{x}_i = x_i + P \in F$. In particular, $\overline{f} = \prod_{i=1}^{n}(T - \overline{x}_i)$. Since $\overline{f}$ is separable, the $\overline{x}_i$ are distinct, and F is a splitting field for $\overline{f}$.

Let $G = \mathrm{Gal}(f/\mathbb{Q})$. Then G maps R to R since it permutes the x_i. Let $H \leq G$ be the stabiliser of P, so $H = \{\sigma \in G \mid \sigma P = P\}$. Since H fixes P, H acts on the quotient $R/P = F$, and it permutes the $\overline{x}_i$ in the same way as it permutes the x_i. In particular, there is an injective homomorphism from H into $\mathrm{Gal}(F/\mathbb{F}_p)$. It now suffices to show that this homomorphism is an isomorphism.

Let $\{P = P_1, P_2, \dots, P_r\}$ be the orbit of P under G, so $P_i = \sigma P$ for some $\sigma \in G$. These are all maximal ideals since P is, and $R/P_i \simeq R/P$ so each R/P_i have p^d elements. The P_i are maximal, so $P_i + P_j = R$ if $i \neq j$. So by the Chinese remainder theorem for rings,

$$R/(P_1 \cap \cdots \cap P_k) \simeq R/P_1 \times \cdots \times R/P_r$$

As $p \in P_1$, $pR \subseteq P_1 \cap \cdots \cap P_r$. So

$$p^N \geq p^M = \left|R/pR\right| \geq \left|R/(P_1 \cap \cdots \cap P_r)\right| = \prod_{i=1}^{r} \left|R/P_i\right| = p^{rd} \implies N \geq rd$$

Now, by the orbit-stabiliser theorem, $r = (G : H) = \frac{N}{|H|}$. Since H injects into $\mathrm{Gal}(F/\mathbb{F}_p)$, we have $|H| \leq d$ with equality if and only if the injection is an isomorphism. So $N \leq rd$, but since $N \geq rd$, we must have $N = rd$, so the injection is an isomorphism, and $H \simeq \mathrm{Gal}(\overline{f}/\mathbb{F}_p)$. □

Corollary. Let $f \in \mathbb{Z}[T]$ be monic and separable with p a prime such that $\overline{f} \in \mathbb{F}_p[T]$ is separable. Consider the factorisation into irreducibles $\overline{f} = g_1 \dots g_r \in \mathbb{F}_p[T]$, where $\deg g_i = n_i$. Then $\mathrm{Gal}(f/\mathbb{Q})$ contains an element of cycle type $(n_1, \dots, n_r)$.

Proof. Combine the previous two theorems. □

Example. Let $f = T^4 - 3T + 1$. Consider $p = 2$. In $\mathbb{F}_2$, $f = T^4 + T + 1$. This does not have a root, and not divisible by $T^2 + T + 1$ which is the only irreducible quadratic, so it is irreducible.

Now, consider $p = 5$. In $\mathbb{F}_5$, $f = (T + 1)(T^3 - T^2 + T + 1)$, which is a factorisation into irreducibles.

By the above corollary, $\mathrm{Gal}(f/\mathbb{Q})$ has a 4-cycle and a 3-cycle. In particular, $12 \mid |\mathrm{Gal}(f/\mathbb{Q})|$, so the group is either all of S_4 or it is A_4, as this is the unique index 2 subgroup of S_4. But 4-cycles are odd, so do not lie in A_4. So $\mathrm{Gal}(f/\mathbb{Q}) = S_4$.

Note that if $\overline{f}$ is separable, $\mathrm{Disc}(\overline{f}) \neq 0$, so $p \nmid \mathrm{Disc}(\overline{f})$ so f is separable. If f is separable, then $\overline{f}$ is separable for all primes but the finite set of primes dividing $\mathrm{Disc}(f)$.

Remark. If $\mathrm{Gal}(f/\mathbb{Q})$ contains an element of cycle type $(n_1, \dots, n_r)$, it can in fact be shown that there exist infinitely many primes p such that $\overline{f}$ factors into irreducibles of degrees $n_1, \dots, n_r$ in $\mathbb{F}_p$. This is known as the Chebotarev density theorem, which is a generalisation of Dirichlet's theorem on primes in arithmetic progression. However, the proof is far outside the scope of this course.

6 Cyclotomic and Kummer extensions

6.1 Primitive roots of unity

Lemma. Let C be a cyclic group of order $n > 1$. Let $a \in \mathbb{Z}$ be coprime with n, also written $(a, n) = 1$. Then the map $[a] : C \to C$ given by $[a](g) = g^a$ is an automorphism of C, and the map $\left(\mathbb{Z}/n\mathbb{Z}\right)^\times \to \mathrm{Aut}(C)$ defined by $a \mapsto [a]$ is an isomorphism.

Proof. $[a]$ is clearly a homomorphism, and since a is coprime to n, it is an automorphism since there exists b such that ab is congruent to 1 modulo n. Hence, there is an injection $\left(\mathbb{Z}/n\mathbb{Z}\right)^\times \to \mathrm{Aut}(C)$ given by $a \mapsto [a]$, and it is a homomorphism. If $\varphi \in \mathrm{Aut}(C)$ and g is a generator for C, $\varphi(g) = g^a$ for some $a \in \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$. So $\varphi = [a]$, and in particular, the map is an isomorphism. □

Let K be a field and $n \geq 1$. We define $\boldsymbol{\mu}_n(K) = \{x \in K \mid x^n = 1\}$ for the group (under multiplication) of nth roots of unity in K. This is a finite subgroup of $K^\times$, hence it is cyclic. The order of any element divides n, so it has order dividing n.

We say that $\zeta \in \boldsymbol{\mu}_n(K)$ is a *primitive* nth root of unity if its order is exactly n. Such a ζ exists if and only if $\boldsymbol{\mu}_n(K)$ has n elements, and then ζ is a generator for the group. In particular, $f = T^n - 1$ has n distinct roots, ζ^i for $i \in \{0, \dots, n-1\}$, and hence it is separable. In general, $f = T^n - 1$ is separable if and only if f is coprime with $f' = nT^{n-1}$, which holds if and only if $n \neq 0$. In this section, we assume that the characteristic of K is zero or is a positive number p that does not divide n, so f is separable.

Let L/K be a splitting field for $T^n - 1$. This is Galois since f is separable, so we can define $G = \mathrm{Gal}(L/K)$. Then $|\boldsymbol{\mu}_n(L)| = n$, and so there exists a primitive nth root of unity $\zeta = \zeta_n \in L$. Such an L is called a *cyclotomic extension*.

Proposition. Let $L = K(\zeta)$. There exists an injective homomorphism $\chi = \chi_n : \mathrm{Gal}(L/K) \to \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$ such that $\chi(\sigma) = a$ implies $\sigma(\zeta) = \zeta^a$. In particular, G is abelian. χ is an isomorphism if and only if G acts transitively on the set of primitive roots of unity in L.

The homomorphism χ is called the *cyclotomic character*.

Proof. $\boldsymbol{\mu}_n(L)$ is cyclic and generated by ζ, so the roots of $T^n - 1$ are the powers of ζ, so $L = K(1, \zeta, \zeta^2, \dots, \zeta^{n-1}) = K(\zeta)$. Consider the action of G on L. This action permutes $\boldsymbol{\mu}_n(L)$, and if $\zeta, \zeta' \in \boldsymbol{\mu}_n(L)$ and $\sigma \in G$, then $\sigma(\zeta\zeta') = \sigma(\zeta)\sigma(\zeta')$, so σ acts as an automorphism of $\boldsymbol{\mu}_n(L)$. $\sigma(\zeta) = \zeta$ if and only if σ is the identity because $L = K(\zeta)$. This gives an injective homomorphism $G \hookrightarrow \mathrm{Aut}\left(\boldsymbol{\mu}_n(L)\right) \simeq \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$.

ζ_n^a is primitive if and only if a is coprime to n. Therefore the set of primitive nth roots of unity is $\left\{\zeta^a \,\middle|\, a \in \left(\mathbb{Z}/n\mathbb{Z}\right)^\times\right\}$, which by the previous part, is the orbit of ζ under G. The map is surjective if and only if there is one orbit, so the result follows. □

6.2 Cyclotomic polynomials

Definition. Let K have characteristic zero or a prime p that does not divide n. The *nth cyclotomic polynomial* is

$$\Phi_n(t) = \prod_{a \in \left(\mathbb{Z}/n\mathbb{Z}\right)^\times} (T - \zeta_n^a)$$

in a splitting field L of $T^n - 1$.

This is the polynomial where the roots are the primitive nth roots of unity. As G permutes the primitive nth roots of unity in L, Φ_n has coefficients in $L^G = K$. The last part of the above proposition shows that χ is surjective if and only if $\Phi_n \in K[T]$ is irreducible.

$x \in L$ satisfies $x^n - 1 = 0$ if and only if x is a primitive dth root of unity for some unique $d \mid n$. Hence $T^n - 1 = \prod_{d|n} \Phi_d$, since the sets of roots are equal. In particular, we could have inductively defined the cyclotomic polynomials by $\Phi_n = \frac{T^n - 1}{\prod_{d|n, d \neq n} \Phi_d}$. This shows that the Φ_n do not depend on the choice of field K, since Φ_n is the image in $K[T]$ of a polynomial in $\mathbb{Z}[T]$.

For example, $\Phi_p = \frac{T^p-1}{T-1} = T^{p-1} + T^{p-2} + \cdots + \cdots + T + 1$. We also have $\Phi_1 = T - 1$ and $\Phi_{p^n}(T) = \frac{T^{p^n}-1}{T^{p^{n-1}}-1} = \Phi_p(T^{p^{n-1}})$. We have $\deg \Phi_n = \left|\left(\mathbb{Z}/n\mathbb{Z}\right)^\times\right| = \varphi(n)$ where φ is the Euler totient function.

Theorem (rationals)**.** Let $K = \mathbb{Q}$. Then χ_n is an isomorphism for all $n > 1$. In particular, $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$, and Φ_n is irreducible over $\mathbb{Q}$.

Proof. The statements in the theorem are all equivalent by the previous results, so it suffices to prove that Φ_n is irreducible over $\mathbb{Q}$. If n is prime, we have already proven its irreducibility by Eisenstein's criterion and Gauss' lemma. We can easily extend this to the case where n is a prime power.

Note that χ_n is an isomorphism if for all primes $p \nmid n$, the residue class of $p \in \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$ is in the image of χ, by factorising a as a product of primes if a is coprime to n. Let f be the minimal polynomial of ζ over $\mathbb{Q}$, and let g be the minimal polynomial of ζ^p over $\mathbb{Q}$. If $f = g$, then ζ^p lies in the orbit of $\mathrm{Gal}(L/K)$ on ζ, so p lies in the image of χ as required. Otherwise, f and g are coprime, and they divide $T^n - 1$ so $fg \mid T^n - 1$. As ζ is a root of $g(T^p)$, we have $f \mid g(T^p)$. Reducing modulo p, $\overline{f} \in \mathbb{F}_p[T]$ divides $\overline{g(T^p)} \in \mathbb{F}_p[T]$. But since we are working over $\mathbb{F}_p$, $\overline{g(T^p)} = \overline{g}(T)^p$. Now, $\overline{f}$ and $\overline{g}$ divide $T^n - 1$ in $\mathbb{F}_p[T]$, which is separable because $p \nmid n$. So $\overline{f} \mid \overline{g}^p$, so $\overline{f} \mid \overline{g}$. But then $\overline{f}^2 \mid \overline{f}\overline{g} \mid T^n - 1$, contradicting separability of $T^n - 1$. □

Therefore, the minimal polynomial of $e^{\frac{2\pi i}{n}}$ over $\mathbb{Q}$ is Φ_n.

Theorem (finite fields)**.** Let $K = \mathbb{F}_p$, and let n be coprime to p. Let L be a splitting field for $T^n - 1$. Then χ_n is an isomorphism from $\mathrm{Gal}(L/K)$ to $\langle p \rangle \leq \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$, the subgroup generated by the residue class of p, and $\chi_n(\varphi_p) = p \bmod n$ where φ_p is the Frobenius endomorphism $x \mapsto x^p$, which is a generator of $\mathrm{Gal}(L/K)$. Further, $[L : K] = r$, where r is the order of p modulo n. Finally, φ_p has cycle type $(r, \ldots, r)$ acting as a permutation of the roots of the cyclotomic polynomial Φ_n, which are the primitive nth roots of unity.

Proof. Since $\varphi_p(\zeta) = \zeta^p$ and $L = K(\zeta)$, by definition of χ_n, we have $\chi_n(\varphi_p) = p$, or more precisely, $p \bmod n$. In particular, $\chi_n(G) = \langle p \rangle$, and as this is a Galois extension, $[L : K] = |G| = |\langle g \rangle| = r$. For the last part, notice that if a and n are coprime, $\varphi_p^k(\zeta^a) = \zeta^a$ holds if and only if $\varphi_p^k(\zeta) = \zeta$, or equivalently, $r \mid k$. So the orbits of φ_p on the set $\{\zeta_n^a \mid (a, n) = 1\}$, which is the set of roots of Φ_n, all have length r. □

Remark. This almost gives another proof of the irreducibility of the cyclotomic polynomials Φ_n over $\mathbb{Q}$. By reduction modulo p, $\mathrm{Gal}(\Phi_n/\mathbb{Q})$ contains $\mathrm{Gal}(\Phi_n/\mathbb{F}_p)$ as a subgroup, up to conjugacy by elements of $S_{\varphi(n)}$. It is not difficult to show that in fact $\chi_n(\mathrm{Gal}(\Phi_n/\mathbb{Q})) \supseteq \chi_n(\mathrm{Gal}(\Phi_n/\mathbb{F}_p)) = \langle p \rangle$. As this holds for all primes p not dividing n, $\chi_n(\mathrm{Gal}(\Phi_n/\mathbb{Q})) = \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$.

Remark. The last part of the above theorem implies that over $\mathbb{F}_p$, the cyclotomic polynomial Φ_n factors as a product of irreducibles of degree r. This depends only on the value of p modulo n. In general, for a polynomial with integer coefficients $f \in \mathbb{Z}[T]$, its factorisation modulo p does not follow an obvious pattern.

Answering this question is part of the Langlands programme, a large area of research in modern number theory. The case where there is such a congruence pattern turns out to be when $\mathrm{Gal}(f/\mathbb{Q})$ is abelian. This study is known as class field theory, which is studied in Part III.

6.3 Quadratic reciprocity

The following theorem is from Part II Number Theory. This theorem has several hundred proofs, and this particular one follows from the above theory on cyclotomic polynomials.

Let p be an odd prime and a an integer coprime to p. Then the *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} +1 & \text{if } a \text{ is a square mod } p \\ -1 & \text{otherwise} \end{cases}$$

Euler's formula for the Legendre symbol is

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p$$

Let q be another odd prime, and consider the case $n = q$ in the above discussion, so $L = K(\zeta_q)$ is a splitting field for $f = T^q - 1 = (T-1)\Phi_q$. On roots of f in L, the Frobenius map φ_p has cycle type $(1, r, \dots, r)$. There are $\frac{q-1}{r}$-many r-cycles. The sign of the permutation φ_p is $(-1)^{(r-1)\frac{q-1}{r}} = (-1)^{\frac{q-1}{r}}$ since q is odd. Note that $2 \mid \frac{q-1}{r}$ holds if and only if $r \mid \frac{q-1}{2}$, or equivalently, $p^{\frac{q-1}{2}} \equiv 1 \bmod 2$. This is in the form of Euler's formula for the Legendre symbol. So the sign of φ_p is exactly $\left(\frac{p}{q}\right)$.

Since $G = \langle \varphi_p \rangle$, the sign of φ_p is $+1$ if and only if $G \subseteq A_q$ since $q = \deg f$. This holds if and only if $\mathrm{Disc}(f)$ is a square in $\mathbb{F}_p$.

Lemma. Let $f = \prod (T - x_i)$ over some field. Then $\mathrm{Disc}(f) = (-1)^{\frac{d(d-1)}{2}} \prod f'(x_i)$. where $d = \deg f$.

This lemma can be shown directly from the definition of the discriminant. We use the above lemma with $f = T^q - 1 = \prod_{a=0}^{q-1}(T - \zeta_q^n)$ and $f' = qT^{q-1}$ to find

$$\mathrm{Disc}(f) = (-1)^{\frac{q(q-1)}{2}} \prod_{a=0}^{q-1} q\zeta_q^{a(q-1)} = (-1)^{\frac{q-1}{2}} q^q \zeta_q^{(q-1)\frac{q(q-1)}{2}} = (-1)^{\frac{q-1}{2}} q^q$$

since q is odd. Hence, by the fact that $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$,

$$\left(\frac{p}{q}\right) = \left(\frac{\mathrm{Disc}(f)}{p}\right) = \left(\frac{(-1)^{\frac{q-1}{2}} q}{p}\right) = \left(\frac{q}{p}\right)(-1)^{\frac{(p-1)(q-1)}{4}}$$

which is the quadratic reciprocity law.

6.4 Construction of regular polygons

Lemma. If m is a positive integer such that $2^m + 1$ is prime, then m is a power of two.

Proof. If q is odd, $2^{qr} + 1 = (2^r + 1)(2^{qr-r} - 2^{qr-2r} + \cdots + 1)$, which is a nontrivial factorisation. □

Ruler and compass construction of a regular n-gon for $n \geq 3$ is equivalent to constructing the real number $\cos\left(\frac{2\pi}{n}\right)$.

Theorem (Gauss)**.** A regular n-gon is contructible if and only if n is a power of two multiplied by a product of distinct primes of the form $2^{2^k} + 1$.

Remark. Let $F_k = 2^{2^k} + 1$ be the kth *Fermat number*. $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$ are all prime. Fermat conjectured that all F_k are prime. This is false; Euler proved that $F_5 = 641 \cdot 6700417$. Many Fermat numbers are known to be composite, and no more have been found to be prime.

Proof. Recall that a real number $x \in \mathbb{R}$ is constructible if and only if there is a sequence of fields $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_n$ such that $x \in K_n$ and $[K_{i+1} : K_i] = 2$. In particular, if x is constructible, $[\mathbb{Q}(x) : \mathbb{Q}] = \deg_{\mathbb{Q}}(x)$ is a power of two. Note that

$$x = \cos\left(\frac{2\pi}{n}\right) = \frac{1}{2}(\zeta_n + \zeta_n^{-1}) \implies \zeta_n^2 - 2x\zeta_n + 1 = 0$$

Since $x \in \mathbb{R}$ and $\zeta_n \notin \mathbb{R}$ (for $n \geq 3$), $[\mathbb{Q}(\zeta_n) : \mathbb{Q}(x)] = 2$. If x is constructible, then $[\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ is a power of two. But $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$.

Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorisation of n. Then $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \prod_{i=1}^r p_i^{e_i - 1}(p - 1)$. This is a power of two if and only if for all odd p_i, we have $e_i = 1$ and $p_i - 1$ is a power of two. By the previous lemma, $\varphi(n)$ is a power of two if and only if n is of the required form.

Now suppose n is of the required form, so $\varphi(n) = 2^m$. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is Galois, with Galois group $G \simeq \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$, which has 2^m elements. There exist subgroups $G = H_0 \supset H_1 \supset \cdots \supset H_m = 1$ such that $[H_i : H_{i+1}] = 2$. Indeed, as $2 \mid 2^m$, by Cauchy's theorem there exists an element $\sigma \in G$ of order 2, assuming G is not the trivial group. Take $H_{m-1} = \langle\sigma\rangle$, and then consider $G/\langle\sigma\rangle$, which contains a subgroup of order 2 by the same argument; we can proceed inductively. Then the tower of fixed fields $K_i = \mathbb{Q}(\zeta_n)^{H_i}$ is a tower of quadratic extensions by the Galois correspondence. □

6.5 Kummer extensions

Theorem (linear independence of field embeddings)**.** Let K, L be fields. Let $\sigma_1, \ldots, \sigma_n : K \to L$ be distinct field homomorphisms. Let $y_1, \ldots, y_n \in L$ be such that for all $x \in K^\times$, $y_1\sigma_1(x) + \cdots + y_n\sigma_n(x) = 0$. Then all $y_i = 0$. In other words, $\sigma_1, \ldots, \sigma_n$ are L-linearly independent elements of the set of functions $K \to L$, considered as an L-vector space.

This is a special case, using $G = K^\times$, of the following theorem.

Theorem (linear independence of characters)**.** Let G be a group and L be a field. Let $\sigma_1, \dots, \sigma_n : G \to L^\times$ be distinct group homomorphisms. Then $\sigma_1, \dots, \sigma_n$ are L-linearly independent elements.

Proof. We use induction on n. If $n = 1$, the result is clear. Suppose $n > 1$. Let $y_1, \dots, y_n \in L$ be such that for all $g \in G$, $y_1\sigma_1(g) + \cdots + y_n\sigma_n(g) = 0$. Since the homomorphisms are distinct, there is an element $h \in G$ such that $\sigma_1(h) \neq \sigma_n(h)$. The σ_i are homomorphisms, so

$$y_1\sigma_1(hg) + \cdots + y_n\sigma_n(hg) = y_1\sigma_1(h)\sigma_1(g) + \cdots + y_n\sigma_n(h)\sigma_n(g) = 0$$

Multiplying the original expression in g by $\sigma_n(h)$ and subtracting,

$$y_1'\sigma_1(g) + \cdots + y_{n-1}'\sigma_{n-1}(g) = 0; \quad y_i' = y_i(\sigma_i(h) - \sigma_n(h))$$

By induction, all $y_i' = 0$. But $\sigma_1(h) \neq \sigma_n(h)$, so $y_1 = 0$. So the original equation $y_1\sigma_1(g) + \cdots + y_n\sigma_n(g) = 0$ can be simplified into $y_2\sigma_2(g) + \cdots + y_n\sigma_n(g) = 0$, so again by induction, all y_i are zero. □

We now consider extensions of the form $L = K(x)$ for $x^n = a \in K$. The special case $a = 1$ gives the cyclotomic extensions. These extensions are not necessarily Galois; for example, $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not Galois. In this section, let $n > 1$, and $n \neq 0$ in K.

Theorem. Let K be a field that contains a primitive nth root of unity $\zeta = \zeta_n$. Let L/K be a field extension with $L = K(x)$, where $x^n = a \in K^\times$. Then L/K is a splitting field for $f = T^n - a$, and is Galois with cyclic Galois group. $[L : K]$ is the least $m \geq 1$ such that $x^m \in K$.

Proof. Note that $\boldsymbol{\mu}_n(K) = \{\zeta^i \mid 0 \leq i < n\}$ has n elements. Then f has n distinct roots $\zeta^i x$ in L. So L is a splitting field for the separable polynomial f, and in particular, L is a Galois extension.

Let $\sigma \in \mathrm{Gal}(L/K) = G$. Then $f(\sigma(x)) = 0$, so $\sigma(x) = \zeta^i x$ for some i, which is unique modulo n. This induces a map $\theta : G \to \boldsymbol{\mu}_n(K) \simeq \mathbb{Z}/n\mathbb{Z}$, given by $\theta(\sigma) = \frac{\sigma(x)}{x}$ which is equal to ζ^i for some i. We claim this is a homomorphism. Let $\sigma, \tau \in G$. Then since $\zeta \in K$, $\tau(\theta(\sigma)) = \theta(\sigma)$. So

$$\theta(\tau\sigma) = \frac{\tau\sigma(x)}{x} = \tau\left(\frac{\sigma(x)}{x}\right) \cdot \frac{\tau(x)}{x} = \tau(\theta(\sigma)) \cdot \theta(\tau) = \theta(\sigma)\theta(\tau)$$

It is injective, because $\theta(\sigma) = 1$ if and only if $\sigma(x) = x$, so $\sigma = \mathrm{id}$. So G is isomorphic to a subgroup of a cyclic group. Hence it is cyclic.

If $m \geq 1$, since L/K is Galois, $x^m \in K$ if and only if for all $\sigma \in G$, $\sigma(x^m) = x^m$. By the definition of θ, this holds if and only if for all $\sigma \in G$, $\theta(\sigma)^m = 1$. So $|G| = [L : K]$ divides m. So $[L : K]$ must be the least m such that $x^m \in K$, as required. □

Corollary. Let K be a field that contains a primitive nth root of unity $\zeta = \zeta_n$. Let $a \in K^\times$. Then $f = T^n - a$ is irreducible over K if and only if a is not a dth power in K for any $1 \neq d \mid n$.

Proof. Let L be a splitting field for $f = T^n - a$, so $L = K(x)$ for $x^n = a$. Then the minimal polynomial of x divides f. So f is irreducible if and only if $f = m_{x,K}$, or equivalently, $[L : K] = n$.

Suppose $n = md$ for $d \neq 1$. Then a is a dth power in K if and only if $x^m \in K$ since $\zeta_n \in K$. By the above theorem, this holds if and only if $|G| \mid m$. □

Remark. This does not hold if we relax the assumption $\zeta_n \in K$. For example, consider $K = \mathbb{Q}$ and $T^4 + 4$.

Definition. Extensions of the form $L = K(x)$ where $x^n = a \in K$ and $\zeta_n \in K$ are called *Kummer extensions*.

Example. Let $n = 2$ and $\operatorname{char} K \neq 2$. Then $\zeta_2 = -1 \in K$. Then $K(\sqrt{a})/K$ is a quadratic Kummer extension if $a \notin (K^\times)^2$. Conversely, any quadratic extension must be of this form.

Theorem. Let K be a field that contains a primitive nth root of unity $\zeta = \zeta_n$ where $n > 1$. Let L/K be a Galois extension with cyclic Galois group of order n. Then L is a Kummer extension of K.

Proof. Let $\operatorname{Gal}(L/K) = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1}\}$. For $y \in L$, let

$$x = R(y) = y + \zeta^{-1}\sigma(y) + \zeta^{-2}\sigma^2(y) + \cdots + \zeta^{-(n-1)}\sigma^{n-1}(y) = \sum_{j=0}^{n-1} \zeta^{-j}\sigma^j(y) \in L$$

This is known as a *Lagrange resolvent*. Then

$$\sigma(x) = \sum_{j=0}^{n-1} \zeta^{-j}\sigma^{j+1}(y) = \sum_{j=0}^{n} \zeta^{1-j}\sigma^j(y) = \zeta x$$

Hence $\sigma(x^n) = \zeta^n x^n = x^n$, so $x^n \in K$. By the linear independence of field embeddings with $\{\sigma_i\} = \{1, \sigma, \dots, \sigma^{n-1}\}$, there exists y such that $R(y) = x \neq 0$. Now, since $\sigma^i x = \zeta^i x$, the $\sigma^i(x)$ are distinct, and so $\deg_K x = n$. In particular, $[K(x) : K] = n = [L : K]$, so $L = K(x)$. □

Example. Let $L/\mathbb{Q}$ be a Galois extension of degree 3. Since $\zeta_3 \notin \mathbb{Q}$, this is not a Kummer extension.

7 Trace and norm

7.1 Trace and norm

Let L/K be an extension of degree n, so L is a K-vector space of dimension n. Let $x \in L$. Then the map $U_x : L \to L$ defined by $U_x(y) = xy$ is K-linear, as it is L-linear. Since it is a linear map, it has a characteristic polynomial, a determinant, and a trace.

Definition. The *trace* and *norm* of $x \in L$ (relative to the extension L/K) are $\operatorname{Tr}_{L/K}(x) = \operatorname{tr} U_x \in K$ and $N_{L/K}(x) = \det U_x \in K$ respectively. The *characteristic polynomial* of $x \in L$ is $f_{x,L/K} = \det(TI - U_x) \in K[T]$ where I is the identity linear transformation.

We sometimes write tr_K, $\det_K$. Let $e_1, \dots, e_n$ be a basis for L/K. Then U_x can be written as a unique K-valued matrix $A = (a_{ij})$, so $xe_i = \sum_j a_{ji}e_j$. Then $\mathrm{Tr}_{L/K}(x) = \mathrm{tr}(A)$, and so on.

Example. Consider the quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ with the basis $1, \sqrt{d}$. Let $x = a + b\sqrt{d}$. Since $x \cdot 1 = a + b\sqrt{d}$ and $x \cdot \sqrt{d} = bd + a\sqrt{d}$,

$$A = \begin{pmatrix} a & bd \\ b & a \end{pmatrix}$$

Hence $\mathrm{Tr}_{L/K}(x) = 2a$ and $N_{L/K}(x) = a^2 - b^2 d$.

Example. Consider $\mathbb{C}/\mathbb{R}$ with the basis $1, i$. Then the matrix of U_{x+iy} is

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

which is the usual encoding of complex numbers as 2×2 real matrices. Note the similarity between this matrix and the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Lemma. Let $x, y \in L$ and $a \in K$, where $n = [L : K]$. Then,
(i) $\mathrm{Tr}_{L/K}(x + y) = \mathrm{Tr}_{L/K}(x) + \mathrm{Tr}_{L/K}(y)$;
(ii) $N_{L/K}(xy) = N_{L/K}(x)N_{L/K}(y)$;
(iii) $N_{L/K}(x) = 0$ if and only if $x = 0$;
(iv) $\mathrm{Tr}_{L/K}(1) = n$ and $N_{L/K}(1) = 1$;
(v) $\mathrm{Tr}_{L/K}(ax) = a\,\mathrm{Tr}_{L/K}(x)$ and $N_{L/K}(ax) = a^n N_{L/K}(x)$.
In particular, $\mathrm{Tr}_{L/K}$ is K-linear and $N_{L/K} : L^\times \to K^\times$ is a homomorphism.

Proof. For part (iii), $N_{L/K}(x) = \det(U_x) \neq 0$ if and only if U_x is invertible. But this holds if and only if x is nonzero because L is a field. The other results follow from the laws of linear transformations. □

7.2 Formulae and applications

Theorem. Let $M/L/K$ be a tower of finite extensions. Then, for all $x \in M$,

$$\mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)) = \mathrm{Tr}_{M/K}(x); \quad N_{L/K}(N_{M/L}(x)) = N_{M/K}(x)$$

Proof. We prove the theorem for the trace; we will not need the result for the norm. Let $x \in M$. Let $u_1, \dots, u_m$ be a basis for M/L, and let $v_1, \dots, v_n$ be a basis for L/K. Let (a_{ij}) be the matrix of $U_{x,M/L}$, so $(a_{ij}) \in \mathrm{Mat}_{m,m}(L)$. Then $\mathrm{Tr}_{M/L}(x) = \sum_{i=1}^m a_{ii}$. For each (i, j), let the matrix of $U_{a_{ij}}$ be $A_{ij} \in \mathrm{Mat}_{n,n}(K)$. Then, $\mathrm{Tr}_{L/K}(\mathrm{Tr}_{M/L}(x)) = \sum_{i=1}^m \mathrm{Tr}_{L/K}(a_{ii}) = \sum_{i=1}^m \mathrm{tr}(A_{ii})$.

Consider the basis $u_1v_1, \dots, u_1v_m, u_2v_1, \dots, u_nv_m$ for M over K. Then the matrix of $U_{x,M/K}$ is the block matrix

$$\begin{pmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{nn} \end{pmatrix}$$

which has trace $\sum_{i=1}^{m} \operatorname{tr}(A_{ii})$ as required. □

Proposition. Let $L = K(x)$, and $f = T^n + c_{n-1}T^{n-1} + \cdots + c_0 \in K[T]$ be the minimal polynomial for x over K. Then $f_{x,L/K} = f$. Further, $\operatorname{Tr}_{L/K}(x) = -c_{n-1}$ and $N_{L/K}(x) = (-1)^n c_0$.

Proof. It suffices to prove the first statement, since the second follows from the fact that the determinant and trace are the given coefficients of the characteristic polynomial for any linear transformation. Consider the basis $1, x, \ldots, x^{n-1}$ for L/K. Then, the matrix of U_x is

$$\begin{pmatrix} 0 & \cdots & & & & -c_0 \\ 1 & 0 & \cdots & & & -c_1 \\ 0 & 1 & 0 & \cdots & & \\ \vdots & 0 & 1 & 0 & \cdots & \\ & \vdots & 0 & 1 & \cdots & \\ & & \vdots & \vdots & \ddots & \\ & & & & & -c_{n-1} \end{pmatrix}$$

which has characteristic polynomial f since it is in rational canonical form. □

Corollary. Let $\operatorname{char} K = p > 0$, and $L = K(x)$ where $x \notin K$ but $x^p \in K$. Then for all $y \in L$, we have $\operatorname{Tr}_{L/K}(y) = 0$ and $N_{L/K}(y) = y^p$.

Proof. Recall that the minimal polynomial of x is $T^p - x^p$, so $[L : K] = p$. Suppose that $y \in K$. By a previous lemma, $\operatorname{Tr}_{L/K}(y) = py = 0$ and $N_{L/K}(y) = y^p$. Otherwise, since $[L : K]$ is prime, $K(y) = L$, and in particular, if $y = \sum a_i x^i$ then $y^p = \left(\sum a_i x^i\right)^p = \sum a_i (x^p)^i \in K$. So the minimal polynomial of y is $T^p - y^p$. Applying the previous proposition, the result follows. □

Proposition. Let L/K be a finite separable extension of degree n. Let $\sigma_1, \ldots, \sigma_n : L \to M$ be the distinct K-homomorphisms of L into a normal closure M for L/K. Then

$$\operatorname{Tr}_{L/K}(x) = \sum_{i=1}^{n} \sigma_i(x); \quad N_{L/K}(x) = \prod_{i=1}^{n} \sigma_i(x); \quad f_{x,L/K} = \prod_{i=1}^{n} (T - \sigma_i(x))$$

Remark. If L/K is finite and Galois, then $\operatorname{Tr}_{L/K}(x) = \sum_{\sigma \in \operatorname{Gal}(L/K)} \sigma(x)$, and the other results are similar.

Proof. It suffices to show the result for the characteristic polynomial. Let $e_1, \ldots, e_n$ be a basis for L/K. Let $P = (\sigma_i(e_j)) \in \operatorname{Mat}_{n,n}(M)$. Recall that the σ_i are linearly independent, so there exist no $y_i \in M$ such that for all j, $\sigma_i(e_j) = 0$. Hence P is nonsingular. Let $A = (a_{ij})$ be the matrix of U_x, so $xe_j = \sum_r a_{rj} e_r$. Applying σ_i, we have

$$\sigma_i(x)\sigma_i(e_j) = \sum_r \sigma_i(e_r) a_{rj}$$

So if S is the diagonal matrix with (i,i)th entry $\sigma_i(x)$, then the given equation can be rewritten as $SP = PA$. Therefore $S = PAP^{-1}$. So S and A are conjugate matrices and hence have the same

characteristic polynomial. We explicitly find that the characteristic polynomial of S is $\prod(T - \sigma_i(x))$ and the characteristic polynomial of A is $f_{x,L/K}$. So they are equal as required. □

Note that since the trace $\mathrm{Tr}_{L/K} : L \to K$ is K-linear, it is either the zero map or surjective.

Theorem. Let L/K be a finite extension. Then, L/K is separable if and only if $\mathrm{Tr}_{L/K}$ is surjective.

Remark. If $\mathrm{char}\, K = 0$, $\mathrm{Tr}_{L/K}(1) = n \neq 0$, so the result holds easily.

Proof. Suppose L/K is separable, and $\sigma_i, \dots, \sigma_n$ are the K-homomorphisms of L into a normal closure M of L/K. Then $\mathrm{Tr}_{L/K}(x) = \sum_{i=1}^{n} \sigma_i(x)$. As the σ_i are linearly independent, there exists x such that $\sum_{i=1}^{n} \sigma_i(x) \neq 0$. So $\mathrm{Tr}_{L/K}(x) \neq 0$, and in particular, it must be surjective as it is K-linear.

Now suppose L/K is inseparable. Then there exists $x \in L$ such that $K(x) \supsetneq K(x^p)$ from example 7 on example sheet 2. As we have shown, $\mathrm{Tr}_{K(x)/K(x^p)} = 0$, so

$$\mathrm{Tr}_{L/K} = \mathrm{Tr}_{L/K(x)} \circ \mathrm{Tr}_{K(x)/K(x^p)} \circ \mathrm{Tr}_{K(x^p)/K} = 0$$

□

Example. Consider the extension of finite fields $\mathbb{F}_{q^n}/\mathbb{F}_q$ for $q = p^r$. This is separable, so there exists $x \in \mathbb{F}_{q^n}$ such that $\mathrm{Tr}(x) = 1$. It is also possible to prove this directly by using the fact that the multiplicative group is cyclic.

Remark. This criterion can be used to give another proof that if M/L and L/K are separable, M/K is also separable.

8 Algebraic closure

8.1 Definition

Definition. A field K is *algebraically closed* if every non-constant polynomial over K splits into linear factors over K.

Remark. An equivalent condition is that the only irreducible polynomials are linear.

Example. The complex numbers $\mathbb{C}$ form an algebraically closed field due to the fundamental theorem of algebra.

Proposition. The following are equivalent.
(i) K is algebraically closed.
(ii) If L/K is a field extension and $x \in L$ is algebraic over K, then $x \in K$.
(iii) If L/K is an algebraic extension, $L = K$.

Proof. (i) implies (ii). Let L/K be a field extension and $x \in L$ algebraic over K. Let f be the minimal polynomial for x over K. Then f is linear, so $x \in K$.

(ii) implies (iii). An extension L/K is algebraic when all $x \in L$ are algebraic over K. So $x \in K$ by (ii).

(iii) implies (i). Let f be an irreducible polynomial, and $L = {}^{K[T]}\!/_{(f)}$, so L/K is a finite algebraic extension. Then $L = K$, so f is linear. □

Proposition. Let L/K be an algebraic extension such that every irreducible polynomial $f \in K[T]$ splits into linear factors in L. Then L is algebraically closed.

Such a field is called an *algebraic closure* of K.

Proof. Let M/L be an extension, and let $x \in M$ be algebraic over L. Then x is algebraic over K. By hypothesis, its minimal polynomial $m_{x,K} \in K[T]$ splits into linear factors over L. So $x \in L$. By criterion (ii) in the previous proposition, L is algebraically closed. □

Remark. An algebraic closure of K is the same as an algebraic extension of K which is algebraically closed.

Corollary. The field $\overline{\mathbb{Q}}$ of algebraic complex numbers is algebraically closed. In particular, $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$.

Proof. We apply the previous result to the extension $\overline{\mathbb{Q}}/\mathbb{Q}$. The extension is algebraic, so it suffices to check that every irreducible polynomial $f \in \mathbb{Q}[T]$ splits into linear factors in $\overline{\mathbb{Q}}$. By the fundamental theorem of algebra, f splits in $\mathbb{C}$. By definition of $\overline{\mathbb{Q}}$, we have $f = \prod(T - x_i)$ where each $x_i \in \overline{\mathbb{Q}}$ as required. □

8.2 Algebraic closures of countable fields

Proposition. Let K be a countable field. Then K has an algebraic closure.

Proof. If K is a countable field, then $K[T]$ is a countable ring. We will enumerate the monic irreducible polynomials $f_i \in K[T]$ for $i \geq 1$. Let $L_0 = K$, and inductively define L_i to be a splitting field for f_i over L_{i-1}.

One can perform this in such a way that no choices need to be made in the construction of the splitting fields. We may also assume that $L_{i-1} \subseteq L_i$ for each $i \geq 1$, because if $\sigma\colon L_{i-1} \to L_i$ is the extension, we can replace L_i with $L_{i-1} \sqcup (L_i \setminus \sigma(L_{i-1}))$. Let $L = \bigcup L_i$ be their union. By construction, every f_i splits in L, so L is an algebraic closure of K. □

Example. $\mathbb{F}_p$ has an algebraic closure.

8.3 Zorn's lemma

For a general field, we need to apply some set-theoretic machinery.

Definition. A binary relation $\preceq$ on a set S is a *partial order* if it is reflexive, transitive, and

antisymmetric. Explicitly, for all $x, y, z \in S$, we have

$$x \preceq x; \quad x \preceq y, y \preceq z \implies x \preceq z; \quad x \preceq y, y \preceq x \implies z = y$$

We say $(S, \preceq)$ is a *partially ordered set*, or a *poset*. It is *totally ordered* if the order is total; $x \preceq y$ or $y \preceq x$ for all $x, y \in S$.

Definition. Let S be a partially ordered set. A *chain* in S is a totally ordered subset. An *upper bound* for a subset T of S is an element $z \in S$ such that for all $x \in T$, we have $x \preceq z$. A *maximal element* of S is an element $y \in S$ such that for all $x \in S$, $y \preceq x$ implies $y = x$.

If S is totally ordered, S has at most one maximal element.

Lemma (Zorn)**.** Let S be a nonempty partially ordered set. Suppose that every chain in S has an upper bound in S. Then S has a maximal element.

This can be proven using the axiom of choice.

Example. Let V be a vector space over K. Then V has a basis; a set $B \subseteq V$ such that any finite subset of B is linearly independent, and for all $v \in V$, there exists $b_1, \dots, b_k \in B$ and $a_1, \dots, a_k \in K$ such that $v = \sum_{i=1}^{k} a_i b_i$. If $V = \{0\}$, the result is trivial by taking $V = \varnothing$. Otherwise, let S be the set of all subsets $X \subseteq V$ where finite subsets of X are linearly independent. S is ordered by inclusion; this is a partial order. S is nonempty since $V \neq \{0\}$. Each chain $T \subseteq S$ has an upper bound by taking its union $Y = \bigcup_{X \in T} X$. This upper bound indeed lies in S, since we only need to check finite subsets of Y for linear independence. Then by Zorn's lemma, S has a maximal element B, which can be seen to be a basis.

Proposition. Let L/K be an algebraic extension, and let M be algebraically closed. Let $\sigma : K \to M$. Then there exists $\overline{\sigma} : L \to M$ extending σ.

Proof. First, consider the case $L = K(x)$ where x is algebraic over K with minimal polynomial $m_{x,K} = f$. Then $\sigma f \in M[T]$. Since M is algebraically closed, σf splits in M. Therefore there exists such a $\overline{\sigma} : K(x) \to M$ extending σ. We can obtain one homomorphism for each root of σf in M.

Now consider the general case. Suppose $K \subseteq L$ without loss of generality, by replacing K with its image in L. Let

$$S = \left\{ (F, \tau) \,\middle|\, K \subseteq F \subseteq L, \tau : F \to M, \tau\Big|_K = \sigma \right\}$$

This has a partial order given by $(F, \tau) \preceq (F', \tau')$ where $F \subseteq F'$ and $\tau'|_F = \tau$. Therefore, S is a partially ordered set. It contains (K, σ), so it is not empty.

Let $T = (F_i, \tau_i)_{i \in I}$ be a chain in S. If T is empty, we can vacuously upper bound it with (K, σ). Otherwise, we define $F' = \bigcup_{i \in I} F_i$. This is a field since T is a chain; in particular, for all $i, j \in I$, we have either $F_i \subseteq F_j$ or $F_j \subseteq F_i$. Now define $\tau' : F' \to M$ by mapping x to $\tau_i(x)$ where $x \in \tau_i$; this is independent of the choice of i since $\tau_j\big|_{F_i} = \tau_i$ and T is a chain. This is an upper bound in S for the chain.

Then, by Zorn's lemma, S has a maximal element. Let (F, τ) be this maximal element. We will show $F = L$; in this case, $\tau = \overline{\sigma}$ is an extension as required.

Clearly $F \subseteq L$. If $x \in L$, then by the first part applied to $F(x)/F$, we can extend the homomorphism $\tau\colon F \to M$ into a homomorphism $\overline{\tau}\colon F(x) \to M$. Then $(F(x), \overline{\tau}) \in S$, and $(F, \tau) \preceq (F(x), \overline{\tau})$. By maximality, $F(x) = F$, so $x \in F$. Hence $F = L$ as required. □

8.4 Algebraic closures of general fields

One can construct an algebraic closure of a field using Zorn's lemma, obtaining a field that extends all algebraic extensions of a given field. However, difficulties arise since the class of algebraic extensions of a field does not form a set. Zorn's lemma can be utilised inside a suitably well-behaved set, but instead, we will construct the algebraic closure via the maximal ideal theorem.

Theorem (maximal ideal theorem)**.** Let R be a non-zero commutative ring with a 1. Then R has a maximal ideal.

Proof sketch. Let S be the set of all proper ideals $I \lhd R$, partially ordered by inclusion. A maximal ideal is a maximal element of S. We apply Zorn's lemma. Let T be a nonempty chain, since anything is an upper bound for an empty chain. Then $J = \bigcup_{I \in T} I$ is an ideal. As $1 \notin I$ for all $I \in T$, we conclude $1 \notin J$. So J is a proper ideal, and hence is an upper bound. □

Theorem. Let K be a field. Then K has an algebraic closure $\overline{K}$. If $\sigma\colon K \to K'$ is an isomorphism, and $\overline{K}, \overline{K}'$ are any algebraic closures of K, K', then σ extends to an isomorphism $\overline{\sigma}\colon \overline{K} \to \overline{K}'$.

Remark. The extension $\overline{\sigma}$ is not generally unique.

Proof. We begin by proving the existence of the algebraic closure. Let P be the set of monic irreducible polynomials in $K[T]$, and construct K_1 such that every $f \in P$ has a root in K_1. First, we will find a ring in which every $f \in P$ has a root.

Let $R = K\left[\{T_f\}_{f \in P}\right]$ be the set of finite K-linear combinations of monomials $T_{f_1}^{m_1} \dots T_{f_k}^{m_k}$ for $f_i \in P$. Let I be the ideal generated by $f(T_f)$ for each $f \in P$. Now, in R/I, $T_f + I$ is a root of f.

We must check that $I \neq R$. If $I = R$, then in particular $1 \in I$. In other words, for some finite subset $Q \subseteq P$, there exists $r_f \in R$ for $f \in Q$ such that $1 = \sum_{f \in Q} r_f f(T_f)$. Enlarging Q if necessary, we can assume that each r_f is a polynomial in $\{T_g \mid g \in Q\}$. Let L/K be a splitting field for $\prod_{f \in Q} f$, and $a_f \in L$ be a root of f for each $f \in Q$. Consider the homomorphism $\varphi\colon R \to L$ such that $\varphi|_K = \mathrm{id}$ and $\varphi(T_f) = a_f$ for $f \in Q$, and $\varphi(T_f) = 0$ for $f \notin Q$. Then

$$1 = \varphi(1) = \sum_{f \in Q} \varphi(r_f f(T_f)) = \sum_{f \in Q} \varphi(r_f) f(a_f) = 0$$

This is a contradiction, so I is in fact a proper ideal.

By the maximal ideal theorem, the ring R/I has a maximal ideal $\overline{J}$. Equivalently, there exists a maximal ideal J of R containing I, since the ideals of R/I are in bijection with the ideals of R containing I by

the isomorphism theorem. Now let $K_1 = {}^{R}\!/_{J}$. This is a field since J is maximal. Let $x_f = T_f + J \in K_1$, then K_1/K is generated by the x_f, and $f(x_f) = 0$ by construction. So K_1/K is an algebraic extension of K in which every $f \in P$ has a root.

Let P_1 be the set of monic irreducibles in $K_1[T]$. We apply the same procedure to K_1 and P_1 to obtain a field K_2, and so on. We then obtain a tower $K \subseteq K_1 \subseteq K_2 \subseteq \cdots$ such that if $f \in K_n[T]$ is non-constant, it has a root in K_{n+1}.

Now, suppose $f \in K[T]$ is non-constant. Then we can write $f = (T - x_1)f_1$ where $x_1 \in K_1, f_1 \in K_1[T]$, and so on. So f splits in $K_{\deg f - 1}$. Therefore, the union $\bigcup_{n\in\mathbb{N}} K_n$ is algebraically closed, and hence is an algebraic closure of K.

We now prove uniqueness. Let $K \subseteq \overline{K}$ and $K' \subseteq \overline{K}'$ be algebraic closures, and let $\sigma\colon K \to K'$ be an isomorphism. Then by the previous result, as $\overline{K}/K$ is algebraic, σ extends to a homomorphism $\overline{\sigma}\colon \overline{K} \to \overline{K}'$. It suffices to show that σ is an isomorphism. We have $K' \subseteq \sigma(\overline{K}) \subseteq \overline{K}'$, so $\overline{K}'/\sigma(\overline{K})$ is algebraic. $\overline{K}$ is algebraically closed, so $\sigma(\overline{K})$ is also algebraically closed. So $\overline{K}' = \sigma(\overline{K})$ by part (iii) of a previous result. □

9 Solving polynomial equations

9.1 Cubics

Let $f \in K[T]$ be a monic separable cubic. Then $G = \mathrm{Gal}(f/K) \leq S_3$ acting on the roots x_1, x_2, x_3 in a splitting field L of K. If f is reducible, f is either a product of three linear factors, in which case G is trivial, or f is a linear factor multiplied by a quadratic, in which case G is isomorphic to S_2.

Now suppose f is irreducible. We will assume that $\mathrm{char}\, K \neq 2, 3$. We have $G = S_3$ or $G = A_3$. We know that $G = A_3$ if and only if the discriminant $\mathrm{Disc}(f)$ is a square in $K^\times$. In general, the Galois correspondence yields

$$\begin{array}{ccc} L = K(x_1, x_2, x_3) & & \{1\} \\ \big\downarrow{\scriptstyle 3 \text{ if } f \text{ irreducible, else } 1} & & \big\uparrow \\ K_1 = K(\Delta) = L^{G\cap A_3} & & G \cap A_3 \\ \big\downarrow{\scriptstyle 2 \text{ or } 1} & & \big\uparrow \\ K & & G \end{array}$$

Then $K_1 = K(\sqrt{\mathrm{Disc}(f)})$, and $K_1 = L$ if f is reducible.

In the irreducible case, L/K_1 is Galois with $\mathrm{Gal}(L/K_1) \simeq {}^{\mathbb{Z}}\!/_{3\mathbb{Z}}$. Recall that if $\omega \in K_1$ is a primitive third root of unity, then $L = K_1(y)$ where $y^3 \in K_1$, by Kummer theory.

We can compute this y explicitly. Suppose $f = T^3 + bT + c$ without loss of generality. Then $\Delta^2 = -4b^3 - 27c^2$. If $b = 0$, the roots of f are $w^i\sqrt[3]{-c}$, so let y be any of them. In the other case $b \neq 0$, let y be a Lagrange resolvent. If the roots of f in L are x_1, x_2, x_3, take $y = x_1 + \omega^2 x_2 + \omega x_3 = (1-\omega)(x_1 - \omega x_2)$ as $x_1 + x_2 + x_3 = 0$. Then $L(\omega) = K(\Delta, \omega, y)$ if and only if $y \neq 0$, by the proof of the structure of Kummer extensions. Let $y' = x_1 + \omega x_2 + \omega^2 x_3$, then $yy' = -3b \neq 0$ since we are not in characteristic 3. Note that $y + y' = y + y' + x_1 + x_2 + x_3 = 3x_1$. One can calculate $y^3 = \frac{1}{2}\left(-3\sqrt{-3}\Delta + 27c\right)$, so $x_1 = y - \frac{3b}{y}$.

If not, let $L(\omega)$ be the splitting field of $f \cdot (T^3 - 1)$ over K. Then $L(\omega)/K_1(\omega)$ is Galois with Galois group $\mathbb{Z}/3\mathbb{Z}$ as before. So $L(\omega) = K_1(\omega, y)$ where $y^3 \in K_1(\omega)$.

Therefore, in every case, x_i lie in the field obtained by adjoining successive square roots and cube roots to K, since $\omega = \frac{-1+\sqrt{-3}}{2}$. This is a theoretical description of Cardano's solution to the cubic.

9.2 Quartics

Let $f \in K[T]$ be a monic separable quartic, with $\operatorname{char} K \neq 2, 3$. Then $\operatorname{Gal}(f/K) \leq S_4$. Note that S_4 acts on the partitions $(12 \mid 34), (13 \mid 24), (14 \mid 23)$ of $\{1, 2, 3, 4\}$. Then we have a homomorphism $S_4 \to S_3$. The kernel of this homomorphism is the Klein four-group $V = \{e, (12)(34), (13)(24), (14)(23)\} \lhd S_4$. Hence the homomorphism is surjective, as $|V| \cdot |S_3| = |S_4|$.

Let f have splitting field L with (distinct) roots $x_1, \dots, x_4$. Suppose that $x_1 + \cdots + x_4 = 0$ without loss of generality as the characteristic is not 2, so $f = T^4 + aT^2 + bT + c$. Since V is a normal subgroup of S_4, $G \cap V$ is a normal subgroup of G and contains V. In particular, we have a homomorphism $G/G \cap V \hookrightarrow S_4/V \simeq S_3$. But $G/G \cap V = \operatorname{Gal}(M/K)$. So we should be able to write M as the splitting field of a cubic $g \in K[T]$.

Let $y_{12} = x_1 + x_2 = -(x_3 + x_4) = -y_{34}$, and let $y_{13}, y_{24}, y_{14}, y_{23}$ be defined similarly. Note that $G \cap V$ maps y_{12} to y_{12} or $y_{34} = -y_{12}$, and so on. So $y_{12}^2, y_{13}^2, y_{14}^2$ are fixed under $G \cap V$. Hence they lie in $M = L^{G \cap V}$.

Suppose $y_{12}^2 = y_{13}^2$. Then either $y_{12} = y_{13}$, so $x_2 = x_3$, contradicting separability, or $y_{12} = -y_{13}$, so $2x_1 + x_2 + x_3 = 0$, giving $x_1 = x_4$, also contradicting separability. So these are distinct elements of M, and hence are indeed the roots of a separable cubic $g \in K[T]$. This is called the *resolvent cubic*.

$M = L^{G \cap V}$ is a splitting field of g. Note that $x_1 = \frac{1}{2}(y_{12} + y_{13} + y_{14})$ and similar results hold for x_2, x_3, x_4. Hence $L = M(y_{12}, y_{13}, y_{14})$. We can compute $g = (T - y_{12}^2)(T - y_{13}^2)(T - y_{14}^2) = T^3 + 2aT^2 + (a^2 - 4c)T - b^2$. In particular, $y_{12}y_{13}y_{14} = b$, hence we can simplify to $L = M(y_{12}, y_{13})$ where $y_{12}^2, y_{13}^2 \in M$.

In conclusion, we have found a way to solve $f = 0$. First, we solve the resolvent equation $g = 0$, and then we take at most two square roots to obtain the relevant field generators.

9.3 Solubility by radicals

Let $f \in K[T]$ be a monic polynomial in a field K of characteristic zero. To prove that there is no quintic formula, we must first establish a definition of 'formula'. The relevant notion is solubility by radicals.

Definition. An irreducible polynomial $f \in K[T]$ is *soluble by radicals* over K if there exists a sequence of fields $K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m$, with $x \in K_m$ a root of f, and each K_i is obtained from K_{i-1} by adjoining a root, so $K_i = K_{i-1}(y_i)$ where $y_i^{d_i} \in K_{i-1}$.

Remark. This is a generalisation of ruler and compass constructions to permit roots of arbitrary degree.

Note that we can adjoin extra roots if desired. In particular, adjoining roots of unity, f is soluble by radicals over K if there exists $d \geq 1$ and $K = K_0 \subseteq \cdots \subseteq K_m$, such that $x \in K_m$ is a root of f, and

$K_1 = K_0(\zeta_d)$ where ζ_d is a primitive dth root of unity. We can also assume that the other extensions satisfy $K_i = K_{i-1}(y_i)$ for $y_i^d = a_i \in K_{i-1}$. This condition can be easily satisfied by letting d be the least common multiple of the d_i that occurs in the tower of fields.

Note that K_1/K_0 is Galois with abelian Galois group. K_i/K_{i-1} for $i > 1$ is Galois, where the Galois group is a subgroup of $\mathbb{Z}/d\mathbb{Z}$ as it is a Kummer extension.

To obtain all roots of f, we consider a normal closure M of K_m; this will contain a splitting field for f, since it contains one root and f is irreducible. To determine M, let $K_i' \subseteq M$ be a normal closure of K_i for each i. As we are in characteristic zero, an extension is Galois if and only if it is normal. Note that K_1 is Galois, so $K_1 = K_1' = K(\zeta_d)$.

Proposition. $K_i' = K_{i-1}'\left(\left\{\sqrt[d]{\sigma(a_i)} \,\middle|\, \sigma \in \mathrm{Gal}(K_{i-1}'/K)\right\}\right)$.

Proof. Suppose $\sigma \in \mathrm{Gal}(K_{i-1}'/K)$. Then we can lift σ to an element $\overline{\sigma} \in \mathrm{Gal}(K_i'/K)$ such that $\overline{\sigma}|_{K_{i-1}'} = \sigma$. Since K_i'/K is normal, it contains $\overline{\sigma}(y_i)$, and $\overline{\sigma}(y)^d = \sigma(y^d) = \sigma(a_i)$. So the right hand side is contained in K_i'.

It suffices to show the right hand side is a normal extension. It is the splitting field over K_{i-1}' of the polynomial $g_i = \prod_{\sigma \in \mathrm{Gal}(K_{i-1}'/K)}(T^d - \sigma(a_i))$. This has coefficients in K. If K_{i-1}' is the splitting field of some polynomial h_{i-1} over K, then the right hand side is the splitting field of the product $g_i h_{i-1}$ over K. So it is normal. □

Proposition. $\mathrm{Gal}(K_i'/K_{i-1}')$ is abelian.

Proof. This proof is a variant on the proof of a previous theorem. Consider the case $i > 1$. Let $A = \mathrm{Gal}(K_i'/K_{i-1}')$. Let $\tau \in A$ and $\sigma \in \mathrm{Gal}(K_i'/K)$. Then $\tau(\sqrt[d]{\sigma(a_i)}) = \zeta_d^{m_\sigma}\sqrt[d]{\sigma(a_i)}$ where $m_\sigma \in \mathbb{Z}/d\mathbb{Z}$. Hence $\tau \mapsto (m_\sigma) \in \left(\mathbb{Z}/d\mathbb{Z}\right)^r$ is an injective homomorphism, where $r = |\mathrm{Gal}(K_{i-1}'/K)|$.

If $i = 1$, then $K_1 = K(\zeta_d)$. So the Galois group is a subgroup of $\left(\mathbb{Z}/d\mathbb{Z}\right)^\times$, so is abelian. □

Since all of the fields K_i' are normal closures, the N_i are normal subgroups of G.

Definition. A finite group G is *soluble* if there exists a chain of normal subgroups $N_i \trianglelefteq G$ with $G = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = \{1\}$ such that N_i/N_{i+1} is abelian for all i.

Example. Any abelian group is soluble. S_3 is soluble, by considering the chain $S_3 \supset A_3 \supset \{1\}$, as $S_3/A_3 \simeq \mathbb{Z}/2\mathbb{Z}$ and $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$. S_4 is also soluble; the chain $S_4 \supset A_4 \supset V \supset \{1\}$ suffices. Note that $S_4/A_4 \simeq \mathbb{Z}/2\mathbb{Z}$, $A_4/V \simeq \mathbb{Z}/3\mathbb{Z}$, $V \simeq \left(\mathbb{Z}/2\mathbb{Z}\right)^2$.

We have shown that $N_i/N_{i+1} = \mathrm{Gal}(K_i'/K_{i-1}')$ is abelian. Hence $\mathrm{Gal}(M/K)$ is soluble.

Lemma. Every subgroup and quotient of a soluble group is soluble.

Proof. Let $G = N_0 \supset N_1 \supset \dots N_m = \{1\}$, where the quotients N_i/N_{i+1} are abelian. Let $H \leq G$. Then $H \cap N_i \trianglelefteq H$, and there is an injective homomorphism from $H \cap N_i/H \cap N_{i+1}$ to N_i/N_{i+1}. Hence the $H \cap N_i/H \cap N_{i+1}$ are abelian, so H is soluble.

Now let $\pi : G \to \overline{G} = G/H$ for $H \trianglelefteq G$. Then $\pi(N_i) \trianglelefteq \overline{G}$, and N_i/N_{i+1} surjects onto $\pi(N_i)/\pi(N_{i+1})$. □

Theorem (Abel–Ruffini)**.** Let $f \in K[T]$ be soluble by radicals over K. Then $\mathrm{Gal}(f/K)$ is soluble.

Proof. $\mathrm{Gal}(f/K) = \mathrm{Gal}(L/K) \simeq \mathrm{Gal}(M/K)/\mathrm{Gal}(M/L)$. We know that $\mathrm{Gal}(M/K)$ is soluble, so the result follows from the fact that quotients of soluble groups are soluble. □

Remark. One can easily show the converse to this theorem.

Proposition. If $n \geq 5$, then S_n and A_n are insoluble.

Proof. S_n and A_n contain A_5 as a subgroup, so it suffices to show that A_5 is insoluble. A_5 is not abelian, and it is simple, so it is insoluble. □

Corollary. Let $n = \deg f \geq 5$, and $A_n \leq \mathrm{Gal}(f/K)$. Then f is not soluble by radicals over K.

10 Miscellaneous results

10.1 Fundamental theorem of algebra

This subsection is non-examinable. We show that $\mathbb{C}$ is algebraically closed over $\mathbb{Q}$, without using complex analysis. We will only use the following facts:

(i) every polynomial of odd degree over $\mathbb{R}$ has a root, due to the intermediate value theorem;

(ii) every quadratic over $\mathbb{C}$ splits into linear factors, so we can take square roots;

(iii) every finite group G has a subgroup H such that $(G : H)$ is odd and $|H|$ is a power of 2, by Sylow's theorem for $p = 2$;

(iv) if G is a p-group, so $|G| = p^k$ and $k > 0$, then G has a subgroup of index p, since G has a non-trivial centre.

Let $K/\mathbb{C}$ be a finite extension. Let L/K be a normal closure of K over $\mathbb{R}$, so L is a Galois extension of $\mathbb{R}$ containing $\mathbb{C}$. Let $G = \mathrm{Gal}(L/\mathbb{R})$. We will show that $L = \mathbb{C}$.

Let $H \leq G$ be a Sylow 2-subgroup, and consider L^H. We have $[L^H : \mathbb{R}] = (G : H)$, which is odd. So if $x \in L^H$, by (i), its minimal polynomial is linear over $\mathbb{R}$, so $x \in \mathbb{R}$. Hence $L^H = \mathbb{R}$, so $H = G$. So G is a 2-group.

Let $G \supset G_1 = \mathrm{Gal}(L/\mathbb{C})$, and $G_2 \leq G_1$ be a subgroup of index 2, which exists by (iv). Then $[L^{G_2} : \mathbb{C}] = (G_1 : G_2)$, contradicting the fact (ii) that quadratics split in $\mathbb{C}$. So there cannot exist a subgroup of index 2, so $G_1 = \{e\}$, and $L = \mathbb{C}$.

10.2 Artin's theorem on invariants

Theorem (Artin)**.** Let L be a field and $G \leq \mathrm{Aut}(L)$ be a finite subgroup of automorphisms of L. Define $L^G = \{x \in L \mid \forall \sigma \in G,\ \sigma(x) = x\}$. Then L/L^G is finite, and satisfies $[L : L^G] = |G|$.

Remark. Unlike in the Galois correspondence, this theorem does not rely on a field extension, just a single field and a finite group of automorphisms. In particular, we find that L/L^G is finite and Galois, with Galois group G.

Proof. It suffices to show L/L^G is finite, because then we can apply the Galois correspondence to show $[L : L^G] = |G|$. Let $K = L^G$, and let $x \in L$. Then if $\{\sigma_1(x), \dots, \sigma_r(x)\}$ is the orbit of G on x, then x is a root of $f = \prod_{i=1}^{r}(T - \sigma_i(x))$. But $f \in L^G[T] = K[T]$. By construction, f is separable. Hence x is algebraic and separable over K, and $\deg_K x \leq |G|$.

Let $y \in L$ have maximal degree. We claim that $K(y) = L$. If not, there exists $x \in L \setminus K(y)$. By above, x, y are algebraic and separable over K. By the primitive element theorem, there exists $z \in L$ such that $K(x, y) = K(z) \supsetneq K(y)$, so $\deg_K z > \deg_K y$. But y was chosen to have maximal degree, so this is a contradiction. □

Remark. One can prove this theorem directly without appealing to the Galois correspondence or the primitive element theorem. This can then be used as a starting point for Galois theory, which then allows the more complicated theorems to be proven.

There are two common ways to construct finite Galois extensions. The first, studied earlier in the course, involves taking the splitting field of a separable polynomial; this method constructs a larger field from a given base field. Artin's theorem provides another way to construct such extensions, by fixing a large field L and constructing the subfield L^G.

Example. Let $\Bbbk$ be a field, and let $L = \Bbbk(X_1, \dots, X_n)$ be the field of rational functions, defined as the fractions of the polynomial ring $\Bbbk[X_1, \dots, X_n]$. Let $G = S_n$ be the symmetric group permuting the X_i. Then $G \leq \mathrm{Aut}(L)$.

Theorem. Let $\Bbbk$ be a field and let $L = \Bbbk(X_1, \dots, X_n)$. Then $L^G = \Bbbk(s_1, \dots, s_n)$.

Proof. Recall that $\Bbbk[X_1, \dots, X_n]^G = \Bbbk[s_1, \dots, s_n]$ where the s_i are the elementary symmetric polynomials in the X_i, and there are no nontrivial relations between the s_i. In particular, $\Bbbk(s_1, \dots, s_n) \subseteq L^G$.

Conversely, let $\frac{f}{g} \in L^G$ for $f, g \in \Bbbk[X_1, \dots, X_n] = R$. Without loss of generality let f, g be coprime. Then for all $\sigma \in G$, $\frac{f}{g} = \frac{\sigma f}{\sigma g}$. By Gauss' lemma, R is a unique factorisation domain, and the units in R are the nonzero constants $\Bbbk^\times$. Hence $\sigma f = c_\sigma f$ and $\sigma g = c_\sigma g$ where $c_\sigma \in \Bbbk^\times$.

Since G is finite and has order $N = n!$, $f = \sigma^N f = c_\sigma^N f$. So c_σ is an Nth root of unity. Then fg^{N-1}, g^N are invariant under σ, so $fg^{N-1}, g^N \in R^G = \Bbbk[s_1, \dots, s_n]$. So $\frac{f}{g} = \frac{fg^{N-1}}{g^N} \in \Bbbk(s_1, \dots, s_n)$. □

Example. Let $L = \Bbbk(X_1, \dots, X_n)$, and let $K = \Bbbk(s_1, \dots, s_n) = L^G$ where $G = S^n$. Then by Artin's theorem, L/K is a finite Galois extension with Galois group G. Let $f = T^n - s_1 T^{n-1} + \cdots + (-1)^n s_n \in K[T]$. Then in L, $f = \prod_{i=1}^{n}(T - X_i)$. Since the X_i are different, f is separable, and L/K is a splitting field for f. Hence $\mathrm{Gal}(f/K) = S^n$. Informally, the general polynomial of degree n has Galois group S^n. It is not difficult to show that for any finite group G, there exists a Galois extension with Galois group isomorphic to G.

10.3 Other areas of study

This is one of a number of theories in *invariant theory*, in which one considers a ring R and a group $G \leq \mathrm{Aut}(R)$, and study R^G. If R is a polynomial ring $\Bbbk[X_1, \dots, X_n]$ and $G \leq S_n$, then knowing R^G can help with the computation of Galois groups algorithmically. For example, if $G = A_n$, then $\Bbbk[X_1, \dots, X_n]^{A_n} = \Bbbk[s_1, \dots, s_n, \Delta]$ where $\Delta = \prod_{i<j}(X_i - X_j)$, for $\mathrm{char}\,\Bbbk \neq 2$.

Now consider $R = \Bbbk[X_1, X_2]$ and $G = \{1, \sigma\}$ where $\sigma(X_i) = -X_i$. Let $\mathrm{char}\,\Bbbk \neq 2$. Then one can show $R^G = \Bbbk[X_1^2, X_2^2, X_1X_2] = \Bbbk[Y_1, Y_2, Y_3]/(Y_1Y_2 - Y_3^2)$. Geometrically, $\{Y_1Y_2 = Y_3^2\} \subset \mathbb{R}^3$ is a double cone. The point at which the cones meet is known as a singularity; such singularities occur in the study of algebraic geometry.

If K and G are fixed, it is not always the case that there exists a Galois extension L/K such that $\mathrm{Gal}(L/K) = G$. For instance, if K is algebraically closed, it has no nontrivial Galois extensions. If $K = \mathbb{F}_p$, then $\mathrm{Gal}(L/K)$ must be cyclic.

The *inverse Galois problem* asks whether every finite group G is the Galois group of some Galois extension $L/\mathbb{Q}$. This is unsolved in the general case. On the extra example sheet, one shows that every abelian group is in fact the Galois group of some Galois extension $L/\mathbb{Q}$. There is a famous theorem by Shafarevich that every finite soluble group is such a Galois group over $\mathbb{Q}$. This is also known to hold for most finite simple groups; in particular, due to a theorem of John Thompson, the monster group is known to be a Galois group over $\mathbb{Q}$.

Perhaps to solve this problem, it would be better to instead understand $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The inverse Galois problem is equivalent to asking whether every finite group is a quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We may also be interested in finding the representations of this group. This leads to the Langlands programme.