Algebraic Geometry

Cambridge University Mathematical Tripos: Part III

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Contents

1 Introduction

1.1 Course description

The course consists of four parts.

(i) The theory of sheaves on topological spaces.

(ii) The definitions of schemes and morphisms between them.

(iii) Properties of schemes, such as the algebraic geometry analogues of compactness and other similar properties.

(iv) Rapid introduction to the cohomology of sheaves.

1.2 Motivation from moduli theory

In moduli theory, we study families of varieties instead of one at a time. In the extreme, we study all varieties of a given 'type' simultaneously. For now, let

$$\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}/\sim$$

where $\mathbf{x} \sim \lambda\mathbf{x}$ for nonzero $\lambda, \mathbf{x}$. A variety is the vanishing locus $\mathbb{V}(S)$ of a set S of homogeneous polynomials in $n+1$ variables. These are subsets of $\mathbb{P}^n$. We present some examples of moduli.

Example. The set of all lines in $\mathbb{P}^2$. A line in $\mathbb{P}^2$ is given by

$$\{aX_0 + bX_1 + cX_2 = 0\}$$

where not all of a, b, c are zero. The set of all lines in $\mathbb{P}^2$ are given by triples (a, b, c). Note that $(\lambda a, \lambda b, \lambda c)$ gives the same line as (a, b, c), so really lines in $\mathbb{P}^2$ correspond exactly to points in $\mathbb{P}^2$. We call the set of all lines in $\mathbb{P}^2$ the dual space $\mathbb{P}^2_{\text{dual}}$. This property is known as projective duality.

The same logic applies to the set of degree d hypersurfaces in $\mathbb{P}^n$; this space corresponds directly to

$$\mathbb{P}^{\binom{n+d}{d}-1}$$

There is an unfortunate consequence of this method of study. Some polynomials are of the form $f = f_1^2 f_2$ for some non-constant f_1, but then $\mathbb{V}(f) = \mathbb{V}(f_1 f_2)$. For example, $(X_0 + X_1 + X_2)^2 \subseteq \mathbb{P}^2$ is a line not a conic. In particular, the limit of a sequence of conics may not be a conic. The solution is to take the set

$$U_d \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$$

in which $[f] \in U_d$ has no repeated factors. But then, U_d is 'not compact', as some points have been removed.

We will now describe the impact of scheme theory on this situation. Fix some $\mathbb{P}^n$, and we will produce a 'space'

$$\mathrm{Var}(\mathbb{P}^n) \subsetneq \mathrm{Hilb}(\mathbb{P}^n)$$

The set $\mathrm{Var}(\mathbb{P}^n)$ bijects onto the set of varieties of $\mathbb{P}^n$. The set $\mathrm{Hilb}(\mathbb{P}^n)$ bijects onto the set of subschemes of $\mathbb{P}^n$, and is compact in the Euclidean topology. In particular, limits of varieties need not be varieties, but limits of schemes are always schemes. One consequence is that in scheme theory,

$$\mathbb{V}(X_0 + X_1 + X_2), \quad \mathbb{V}((X_0 + X_1 + X_2)^2)$$

are not isomorphic as schemes in $\mathbb{P}^2$.

1.3 Motivation from the Weil conjectures

Fix some homogeneous polynomial $f \in \mathbb{Z}[X_0, \dots, X_{n+1}]$. First, consider

$$X = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$$

and assume that X is smooth. As X is a compact topological space, we can find its Betti numbers $b_0(X), \dots, b_{2n}(X)$, where

$$b_i(X) = \operatorname{rank} H_i(X; \mathbb{Z})$$

In particular, we can find its Euler characteristic.

$$\chi(X) = \sum (-1)^i b_i(X)$$

Second, fix a prime p and let N_m be the number of solutions of f over $\mathbb{F}_{p^m}$. Define the Weil zeta function

$$\zeta(X; t) = \exp\left(\sum_m \frac{N_m}{m} \cdot t^m \right)$$

One of the Weil conjectures states the following.

Theorem (Grothendieck). (i) $\zeta(X; t)$ is a rational function in t, so

$$\zeta(X; t) = \frac{P_X(t)}{Q_X(t)}$$

(ii) Further, $\zeta(X; t)$ can be written as a ratio of the form

$$\frac{P_0(t) P_2(t) \dots P_{2n}(t)}{P_1(t) P_3(t) \dots P_{2n-1}(t)}$$

where

$$\deg P_i(t) = b_i(X)$$

The proof relies fundamentally on scheme theory: we need a space $\mathcal{X}$ that interpolates between the algebraic closure $\overline{\mathbb{F}_p}$ and $\mathbb{C}$.

1.4 Summary of classical algebraic geometry

Let $k = \overline{k}$ be an algebraically closed field. The notation $\mathbb{A}^n_k = \mathbb{A}^n$ denotes affine space of dimension n over the field k. As a set, this is equal to k^n. An *affine variety* is a subset $V \subseteq \mathbb{A}^n$ of the form

$$V = \mathbb{V}(S) = \{x \in \mathbb{A}^n \mid \forall f \in S,\ f(x) = 0\}$$

where $S \subseteq k[X_1, \dots, X_n]$. Note that $\mathbb{V}(S) = \mathbb{V}(I(S))$, where $I(S)$ is the ideal generated by S. By Hilbert's basis theorem, or equivalently the fact that $k[\mathbf{X}]$ is Noetherian, $\mathbb{V}(S)$ is the vanishing locus of a finite set (even a finite subset of S). In fact, $\mathbb{V}(I) = \mathbb{V}\left(\sqrt{I}\right)$ where

$$\sqrt{I} = \{f \in k[\mathbf{X}] \mid \exists n > 0,\ f^n \in I\}$$

Note that $\sqrt{I}$ is an ideal, and is called the *radical ideal* of I. For example, in $k[X]$, if $I = (X^2)$ then $\sqrt{I} = (X)$. Notice that an affine variety is a subset of $\mathbb{A}^n$ for some n, so we have really defined varieties with a chosen n; we have not defined an abstract variety.

A *morphism* between varieties $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ is a set-theoretic map $\varphi : V \to W$ such that if $\varphi(f_1, \dots, f_m)$, each f_i is the restriction of a polynomial in $\{X_1, \dots, X_n\}$ to V. Note that the polynomials f_i are not part of the definition; a given set-theoretic map may be represented by multiple polynomials. This indicates that the ambient spaces $\mathbb{A}^n, \mathbb{A}^m$ are not relevant to this definition. Isomorphisms are those morphisms with two-sided inverses.

The basic correspondence of the theory of algebraic varieties is

$$\frac{\{\text{affine varieties over } k\}}{\text{isomorphism}} \leftrightarrow \{\text{finitely generated } k\text{-algebras without nilpotent elements}\}$$

We explain each direction of the correspondence. Given a variety V representing an isomorphism class of affine varieties over k, we can write V as the vanishing locus of some radical ideal $I \subseteq k[X_1, \dots, X_n]$. We can then produce the finitely generated k-algebra given by the quotient

$$k[X_1, \dots, X_n]/I$$

This is nilpotent-free as I is radical. In reverse, if A is a finitely generated nilpotent-free k-algebra, then by definition we can write A as

$$k[Y_1, \dots, Y_m]/J$$

where J is radical, or at least up to isomorphism. Then we can produce the affine variety $V = \mathbb{V}(J)$. One must show that the choices we made in the above explanation do not matter.

Note that, for example, $k[X]/(X^2)$ has a nilpotent element X. The theory of schemes explains the relevance of these nilpotent elements, but the theory of varieties 'ignores' nilpotent elements.

The algebra associated to V is classically denoted $k[V]$, and is called the *coordinate ring* of V. There is a bijection between morphisms $V \to W$ and k-algebra homomorphisms $k[W] \to k[V]$. In category theoretic terminology, the category whose objects are affine varieties up to isomorphism is equivalent to the category of finitely generated k-algebras up to isomorphism.

Let $V = \mathbb{V}(I) \subseteq \mathbb{A}^n$ be a variety with coordinate ring $k[V]$. The *Zariski* topology on V is defined such that the closed sets are exactly those sets of the form $\mathbb{V}(S)$ where $S \subseteq k[V]$. One can show that this really induces a topology. If $V \cong W$, then V and W are homeomorphic as topological spaces.

Let V be a variety and $k[V]$ be its coordinate ring. For all points $P \in V$, we can produce a homomorphism $\mathrm{ev}_P : k[V] \to k$ mapping f to $f(P)$; one can check that this is well-defined. Note that ev_P is surjective by considering the constant functions. Thus the kernel of ev_P is a maximal ideal $\mathfrak{m}_P$. We thus obtain

$$\{\text{points of } V\} \to \{\text{maximal ideals in } k[V]\}$$

Hilbert's *Nullstellensatz* states, among other things, that this is a bijection.

1.5 Limitations of classical algebraic geometry

The description of varieties given above always retains information about its ambient affine space, so we cannot define an abstract variety. Similarly to manifolds which locally look like vector spaces, we want to consider 'spaces' that locally look like affine varieties. For example, projective space does not live inside an affine space.

Let $I = (X^2 + Y^2 + 1) \subseteq \mathbb{R}[X, Y]$. Observe that $\mathbb{V}(I)$ is empty in $\mathbb{R}^2$, but I is prime and hence radical. Hence the Nullstellensatz fails in this case. It is then natural to ask on which topological space ${}^{\mathbb{R}[X,Y]}\big/_{(X^2+Y^2+1)}$ is naturally the set of functions. Similar questions can be asked about $\mathbb{Z}$ or $\mathbb{Z}[X]$, for example.

Consider $C = \mathbb{V}(Y - X^2) \subseteq \mathbb{A}^2_k$ and $D = \mathbb{V}(Y)$. Then $C \cap D = \mathbb{V}(X^2, Y) = \mathbb{V}(X, Y) = \{(0,0)\}$. If $D_\delta = \mathbb{V}(Y + \delta)$ for $\delta \in k$, $C \cap D_\delta$ is two points unless $\delta = 0$. This breaks a continuity property. Therefore, the intersection of two affine varieties is not naturally an affine variety.

1.6 Spectrum of a ring

Let A be a commutative unital ring.

Definition. The *Zariski spectrum* of A is $\operatorname{Spec} A = \{\mathfrak{p} \trianglelefteq A \text{ prime}\}$.

Remark. Given a ring homomorphism $\varphi : A \to B$, we have an induced map of sets $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ given by $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$, as the preimage of a prime ideal is always prime. Note, however, that this property would fail if we only considered maximal ideals, because the preimage of a maximal ideal need not be maximal.

Given $f \in A$ and a point $\mathfrak{p} \in \operatorname{Spec} A$, we have an induced $\overline{f} \in {}^{A}\big/_{\mathfrak{p}}$ obtained by taking the quotient. We can think of this operation as 'evaluating' an $f \in A$ at a point $\mathfrak{p} \in \operatorname{Spec} A$, with the caveat that the codomain of this evaluation depends on $\mathfrak{p}$.

Example. (i) Let $A = \mathbb{Z}$. Then $\operatorname{Spec} A = \operatorname{Spec} \mathbb{Z}$ is the set $\{(p) \mid p \text{ prime}\} \cup \{(0)\}$. Consider an element of $\mathbb{Z}$, say, 132. Given a prime p, we can 'evaluate it at p', giving $132 \bmod p \in {}^{\mathbb{Z}}\big/_{p\mathbb{Z}}$. Thus $\operatorname{Spec} \mathbb{Z}$ is a space, 132 is a function on $\operatorname{Spec} \mathbb{Z}$, and $132 \bmod p$ is the value of this function at p.

(ii) Let $A = \mathbb{R}[X]$. Then $\operatorname{Spec} A$ is naturally $\mathbb{C}$ modulo complex conjugation, together with the zero ideal.

(iii) If $A = \mathbb{C}[X]$, then $\operatorname{Spec} A$ is naturally $\mathbb{C}$, together with the zero ideal.

Definition. Let $f \in A$. Then we define

$$\mathbb{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f = 0 \bmod \mathfrak{p}, \text{ or equivalently, } f \in \mathfrak{p}\} \subseteq \operatorname{Spec} A$$

Similarly, for $J \trianglelefteq A$ an ideal,

$$\mathbb{V}(J) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \forall f \in J,\ f \in \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid J \subseteq \mathfrak{p}\}$$

Proposition. The sets $\mathbb{V}(J) \subseteq \operatorname{Spec} A$ ranging over all ideals $J \trianglelefteq A$ form the closed sets of a topology.

This topology is called the *Zariski topology* on A.

Proof. We have $\varnothing = \mathbb{V}(1)$ and $\operatorname{Spec} A = \mathbb{V}(0)$, so they are closed. Note that

$$\mathbb{V}\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha \mathbb{V}(I_\alpha)$$

It remains to show $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) = \mathbb{V}(I_1 \cap I_2)$. The containment $\mathbb{V}(I_1) \cup \mathbb{V}(I_2) \subseteq \mathbb{V}(I_1 \cap I_2)$ is clear. Conversely, note $I_1 I_2 \subseteq I_1 \cap I_2$. If $I_1 \cap I_2 \subseteq \mathfrak{p}$, then by primality of $\mathfrak{p}$, either $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$. $\square$

Example. Consider $\operatorname{Spec} \mathbb{C}[x, y]$. The point $(0) \in \operatorname{Spec} \mathbb{C}[x, y]$ is dense in the Zariski topology, so $\overline{\{(0)\}} = \operatorname{Spec} \mathbb{C}[x, y]$. This is because all prime ideals in integral domains contain the zero ideal. (0) is sometimes called the *generic point*.

Consider the prime ideal $(y^2 - x^3)$, and consider a maximal ideal $\mathfrak{m}_{a,b} = (x - a, y - b)$ corresponding to the point (a, b). Then one can show that

$$\mathfrak{m}_{a,b} \in \overline{\{(y^2 - x^3)\}} \iff b^2 = a^3$$

In general, points are not closed.

1.7 Distinguished opens and localisation

Definition. Let $f \in A$. Define the *distinguished open* corresponding to f to be

$$U_f = \operatorname{Spec} A \setminus \mathbb{V}(f)$$

Example. (i) Let $A = \mathbb{C}[x]$, and recall that $\operatorname{Spec} A$ is $\mathbb{C} \cup \{(0)\}$, where the complex number a represents the maximal ideal $(x - a)$. Let $f = x$ and consider

$$\mathbb{V}(x) = \{\mathfrak{p} \mid x \in \mathfrak{p}\} = \{(x)\}$$

Hence $U_x = \operatorname{Spec} A \setminus \{(x)\}$, which is $\operatorname{Spec} A$ without the complex number 0.

(ii) More generally, suppose we fix $a_1, \dots, a_r \in \mathbb{C}$. Then

$$U = \operatorname{Spec} A \setminus \{(x - a_i)\}_{i=1}^r = U_f; \quad f = \prod_{i=1}^r (x - a_i)$$

Lemma. The distinguished opens U_f, taken over all $f \in A$, form a basis for the Zariski topology on $\operatorname{Spec} A$; that is, every open set in $\operatorname{Spec} A$ is a union of some collection of the U_f.

Proof. Let $U = \operatorname{Spec} A \setminus \mathbb{V}(J)$ be an open set. Then

$$\mathbb{V}(J) = \mathbb{V}\left(\sum_{f \in J} (f)\right) = \bigcap_{f \in J} \mathbb{V}(f)$$

So

$$U = \bigcup_{f \in J} U_f$$

$\square$

Definition. Let $f \in A$. The *localisation* of A at f is

$$A_f = {}^{A[x]}\!\big/_{(xf-1)}$$

Informally, we adjoin $\frac{1}{f}$ to A.

Lemma. The distinguished open $U_f \subseteq \operatorname{Spec} A$ is naturally homeomorphic to $\operatorname{Spec} A_f$ via the ring homomorphism $j : A \to A_f$.

Proof. We will exhibit a bijection between the prime ideals in A_f and the prime ideals in A that do not contain f, producing a homeomorphism as required. Given $\mathfrak{q} \subseteq A_f$ prime, its contraction $j^{-1}(\mathfrak{q})$ is a prime ideal in A.

Now suppose $\mathfrak{p} \subseteq A$ is a prime ideal, and let $\mathfrak{p}_f = j(\mathfrak{p}) \cdot A_f$. We show that $j(\mathfrak{p}) \cdot A_f$ is a prime ideal if and only if $f \notin \mathfrak{p}$, giving the result. If $f \in \mathfrak{p}$, then the unit f lies in $\mathfrak{p}_f$. Thus $\mathfrak{p}_f = (1)$, so is not prime. If $f \notin \mathfrak{p}$, observe that

$$ {}^{A_f}\!\big/_{\mathfrak{p}_f} \cong \left({}^{A}\!\big/_{\mathfrak{p}}\right)_{\overline{f}}; \quad \overline{f} = f + \mathfrak{p}$$

But then,

$$\left({}^{A}\!\big/_{\mathfrak{p}}\right)_{\overline{f}} \subseteq FF\left({}^{A}\!\big/_{\mathfrak{p}}\right)$$

Since $\mathfrak{p}$ is prime, ${}^{A}\!\big/_{\mathfrak{p}}$ is an integral domain, so its fraction field is well-defined. So $\mathfrak{p}_f$ is a prime ideal. One can then check that our two constructions are inverse to each other, providing a bijection between prime ideals as required. $\square$

Remark. (i) $U_f \cap U_g = U_{fg}$. Indeed, if $\mathfrak{p} \in U_{fg}$, then $fg \notin \mathfrak{p}$, so clearly neither f nor g can lie in $\mathfrak{p}$; conversely, if $\mathfrak{p} \in U_f \cap U_g$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, so by primality, $fg \notin \mathfrak{p}$.

(ii) The distinguished opens U_f do not uniquely define an element $f \in A$. For instance, one can easily show that $U_{f^n} = U_f$ for all $n \geq 1$, using the properties of prime ideals.

(iii) In line with (ii), the localisations A_f and A_{f^n} are homeomorphic in a natural way. If

$$A_f = {}^{A[x]}\!\big/_{(xf-1)}; \quad A_{f^n} = {}^{A[y]}\!\big/_{(yf^n-1)}$$

then consider the inverse A-algebra homomorphisms given by

$$x \mapsto f^{n-1}y; \quad y \mapsto x^n$$

Informally, we map $\frac{1}{f}$ to $f^{n-1}\frac{1}{f^n}$, and $\frac{1}{f^n}$ to $\left(\frac{1}{f}\right)^n$.

(iv) The containment $U_f \subseteq U_g$ holds if and only if f^n is a multiple of g for some $n \geq 1$. First, if f^n is a multiple of g, then the claim holds by (i). Now suppose $U_f \subseteq U_g$, so $\mathbb{V}(f) \supseteq \mathbb{V}(g)$. Hence, all prime ideals that contain g also contain f. But since

$$\sqrt{I} = \bigcap_{\mathfrak{p} \text{ prime} \supseteq I} \mathfrak{p}$$

we must have

$$\sqrt{(f)} \supseteq \sqrt{(g)}$$

giving the result.

Remark. For a fixed ring A, we have made an assignment

$$\{\text{distinguished opens in } \operatorname{Spec} A\} \to \mathbf{Rng}$$

given by $U_f \mapsto A_f$, where **Rng** denotes the class of rings. This association is functorial: if $U_{f_1} \subseteq U_{f_2}$, there is a natural map $A_{f_2} \to A_{f_1}$, which should be viewed as the restriction map from functions defined on U_{f_2} to those defined on f_1. This produces a *sheaf*; we now explore these in more generality.

2 Sheaves

2.1 Presheaves

Definition. Let X be a topological space. Let $\operatorname{Open} X$ be the set of open sets on X, and **AbGp** be the class of abelian groups. A *presheaf* $\mathcal{F}$ on X of abelian groups is an association

$$\operatorname{Open} X \to \mathbf{AbGp}$$

and for open sets $U \subseteq V$, a *restriction map*

$$\operatorname{res}^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that

$$\operatorname{res}^U_U = \mathrm{id}; \quad \operatorname{res}^V_U \circ \operatorname{res}^W_V = \operatorname{res}^W_U$$

Example. For any topological space X, the presheaf of real-valued continuous functions on X is defined by

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$$

and

$$\operatorname{res}^V_U(f) = f\Big|_U$$

One can also define presheaves of rings, sets, or other objects by simply replacing the words 'abelian groups' in the definition.

Definition. A *morphism* φ of presheaves $\mathcal{F}, \mathcal{G}$ on X is, for each open set U in X, a homomorphism

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

such that

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "\mathrm{res}^V_U"] \arrow[d, "\varphi(U)"'] & \mathcal{F}V \arrow[d, "\varphi(V)"] \\ \mathcal{G}U \arrow[r, "\mathrm{res}^V_U"'] & \mathcal{G}V \end{tikzcd}$$

commutes.

Remark. A presheaf on a topological space X is just a functor $(\mathrm{Open}\, X)^{\mathrm{op}} \to \mathbf{AbGp}$, where $\mathbf{AbGp}$ is the category of abelian groups, and $\mathrm{Open}\, X$ is the category where the objects are the open sets in X, and there is a morphism $U \to V$ if and only if $U \subseteq V$. A morphism of presheaves is just a natural transformation between two such functors. Replacing $\mathbf{AbGp}$ with an arbitrary category $\mathcal{C}$, we can define presheaves on X of objects in $\mathcal{C}$.

Definition. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves is *injective* (respectively *surjective*) if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (respectively surjective) for all open sets U of X.

2.2 Sheaves

Definition. A *sheaf* on X is a presheaf $\mathcal{F}$ on X such that

(i) if $U \subseteq X$ is open and $\{U_i\}$ is an open cover of U, then for $s \in \mathcal{F}(U)$, if $\mathrm{res}^U_{U_i} s = 0$ for all i, then $s = 0$; and

(ii) if $U, \{U_i\}$ are as in (i), given $s_i \in \mathcal{F}(U_i)$ such that $\mathrm{res}^{U_i}_{U_i \cap U_j} s_i = \mathrm{res}^{U_j}_{U_i \cap U_j} s_j$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $\mathrm{res}^U_{U_i} s = s_i$.

Remark. These two axioms imply that $\mathcal{F}(\varnothing) = 0$.

A morphism of sheaves is a morphism of the underlying presheaves.

Example. (i) Let X be a topological space. Then the presheaf $\mathcal{F}$ given by

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$$

is a sheaf.

(ii) Let $X = \mathbb{C}$ with the usual Euclidean topology, and let

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \mid f \text{ bounded and holomorphic}\}$$

Then $\mathcal{F}$ is not a sheaf, because the functions id_U on bounded open sets U do not glue together to a bounded holomorphic function on all of $\mathbb{C}$. This is a failure of locality in our definition of $\mathcal{F}$; whether f is bounded is a global condition.

(iii) Let G be a group and set $\mathcal{F}(U) = G$, giving the constant presheaf. This is not in general a sheaf. For example, if U_1, U_2 are disjoint, then $\mathcal{F}(U_1 \cup U_2) \simeq G \times G$. Instead, we can give G the discrete topology, and define

$$\mathcal{F}(U) = \{f : U \to G \mid f \text{ continuous}\} = \{f : U \to G \mid f \text{ locally constant}\}$$

This is now a sheaf, called the constant sheaf.

(iv) Let V be an irreducible variety over k. Let

$$\mathcal{O}_V(U) = \{f \in k(V) \mid \forall p \in U,\ f \text{ regular at } p\}$$

where a function f is regular at p precisely if it can be represented as a quotient $\frac{g}{h}$ in a neighbourhood of p on which h is nonzero. This is called the *structure sheaf* of V; it is a sheaf since regularity is a local condition.

2.3 Stalks

Definition. Let $\mathcal{F}$ be a presheaf. A *section* of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$.

Definition. Let $p \in X$, and $\mathcal{F}$ a presheaf on X. Then the *stalk* of $\mathcal{F}$ at p is

$$\mathcal{F}_p = {\{(U,s) \mid s \in \mathcal{F}(U), p \in U\}}\big/{\sim}$$

where

$$(U,s) \sim (V,s') \iff \exists W \subseteq U \cap V \text{ open with } p \in W \text{ such that } \operatorname{res}^U_W s = \operatorname{res}^V_W s'$$

Elements of $\mathcal{F}_p$ are called *germs*.

Example. Let $\mathbb{A}^1$ be the affine line, and let $\mathcal{O}_{\mathbb{A}^1}$ be the sheaf of regular functions. Its stalk at 0 is

$$\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \,\middle|\, g(0) \neq 0 \right\} = k[t]_{(t)}$$

Proposition. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Suppose that for all $p \in X$, the induced map $f_p : \mathcal{F}_p \to \mathcal{G}_p$ given by

$$f_p((U,s)) = (U, f_U(s))$$

is an isomorphism. Then f is an isomorphism.

Proof. We will show that $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ are isomorphisms for each U, then define f^{-1} by $(f^{-1})_U = (f_U)^{-1}$.

To show f_U is injective, consider $s \in \mathcal{F}(U)$ with $f_U(s) = 0$. Since f_p is injective, $(U,s) = 0$ in $\mathcal{F}_p$ for every point $p \in U$. Thus for each $p \in U$, there exists an open neighbourhood $U_p \subseteq U$ such that $\operatorname{res}^U_{U_p} s = 0$. The sets $\{U_p \mid p \in U\}$ cover U, so as $\mathcal{F}$ is a sheaf, $s = 0$.

To show f_U is surjective, let $t \in \mathcal{G}(U)$. For each $p \in U$, there is an element $(U_p, s_p) \in \mathcal{F}_p$ such that $f_p((U_p, s_p)) = (U, t) \in \mathcal{G}_p$. By shrinking U_p if necessary, we can assume $f_{U_p}(s_p) = \operatorname{res}^U_{U_p} t$. For points $p, p' \in U$,

$$f_{U_p \cap U_{p'}}\left(\operatorname{res}^{U_p}_{U_p \cap U_{p'}} s - \operatorname{res}^{U_{p'}}_{U_p \cap U_{p'}} s'\right) = \operatorname{res}^U_{U_p \cap U_{p'}} t - \operatorname{res}^U_{U_p \cap U_{p'}} t = 0$$

Thus

$$\operatorname{res}^{U_p}_{U_p \cap U_{p'}} s - \operatorname{res}^{U_{p'}}_{U_p \cap U_{p'}} s' = 0$$

by injectivity of $f_{U_p \cap U_{p'}}$. So there exists a section s of $\mathcal{F}$ over U such that $\operatorname{res}^U_{U_p} s = s_p$. We now show $f_U(s) = t$. Consider

$$\operatorname{res}^U_{U_p} f_U(s) = f_{U_p}\left(\operatorname{res}^U_{U_p} s\right) = f_{U_p}(s_p) = \operatorname{res}^U_{U_p} t$$

Thus $f_U(s) = t$. □

Remark. (i) Consider the map $\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$ given by $s \mapsto ((U, s))_{p \in U}$. This is injective by the first sheaf axiom.

(ii) Given two morphisms of sheaves $\varphi, \psi : \mathcal{F} \rightrightarrows \mathcal{G}$ with $\varphi_p = \psi_p$ for all $p \in X$, we have $\varphi = \psi$.

2.4 Sheafification

Definition. Let $\mathcal{F}$ be a presheaf on X. Then a morphism $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ to a sheaf $\mathcal{F}^{\mathrm{sh}}$ is a *sheafification* if for any map $\varphi : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, φ factors uniquely through sh.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\varphi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, dashed] \\ & \mathcal{G} \end{tikzcd}$$

Remark. (i) As this is a definition by a universal property, $\mathcal{F}^{\mathrm{sh}}$ along with the map $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$ are unique up to unique isomorphism if they exist.

(ii) A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism of sheaves $\mathcal{F}^{\mathrm{sh}} \to \mathcal{G}^{\mathrm{sh}}$.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "\varphi"'] & \mathcal{F}^{\mathrm{sh}} \arrow[d, dashed] & \\ & \mathcal{G} \arrow[r, "\mathrm{sh}"'] & \mathcal{G}^{\mathrm{sh}} \end{tikzcd}$$

Proposition. Every presheaf admits a sheafification.

Corollary. The stalks of $\mathcal{F}$ and $\mathcal{F}^{\mathrm{sh}}$ coincide.

Proof. Suppose (U, f) is a germ of $\mathcal{F}^{\mathrm{sh}}$ at $p \in X$. Then $f(p) \in \mathcal{F}_p$ is a germ of $\mathcal{F}$ at p. If $(U, s) \in \mathcal{F}_p$, we can produce the germ $(U, (U, s)_{p \in U})$ of $\mathcal{F}^{\mathrm{sh}}$ at $p \in X$. These are inverse operations, and hence give a bijection of stalks. □

2.5 Kernels and cokernels

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then we can define presheaves $\ker\varphi, \operatorname{coker}\varphi, \operatorname{im}\varphi$ by

$$(\ker\varphi)(U) = \ker\varphi_U$$
$$(\operatorname{coker}\varphi)(U) = \operatorname{coker}\varphi_U$$
$$(\operatorname{im}\varphi)(U) = \operatorname{im}\varphi_U$$

One can check that these are indeed presheaves.

Proposition. The presheaf kernel for a morphism of sheaves is a sheaf.

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, let $U \subseteq X$ be open, and let $\{U_i\}_{i\in I}$ be an open cover of U. Let $f \in (\ker\varphi)(U)$ be such that $\operatorname{res}^U_{U_i} f = 0$ for each f. Then as $f \in \mathcal{F}(U)$, we can use the fact that $\mathcal{F}$ is a sheaf to conclude $f = 0$.

Now suppose $f_i \in (\ker\varphi)(U_i)$ agree on their intersections. Then they can be glued as elements of $\mathcal{F}(U_i)$ into $f \in \mathcal{F}(U)$. As $\varphi_{U_i}(f_i) = 0$ for each $i \in I$,

$$0 = \varphi_{U_i}(\operatorname{res}^U_{U_i} f) = \operatorname{res}^U_{U_i} \varphi_U(f)$$

So as $\mathcal{G}$ is a sheaf, $\varphi_U(f) = 0$ in $\mathcal{G}(U)$. $\square$

However, the presheaf cokernel of a morphism of sheaves is not in general a sheaf.

Example. Consider $X = \mathbb{C}$ with the Euclidean topology, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X under addition. Let $\mathcal{O}_X^\star$ be the sheaf of nowhere vanishing holomorphic functions under multiplication. We have a morphism of sheaves

$$\exp : \mathcal{O}_X \to \mathcal{O}_X^\star$$

given by

$$f \in \mathcal{O}_X(U) \mapsto \exp(f) \in \mathcal{O}_X^\star$$

The kernel of exp is $2\pi i\mathbb{Z}$, where $\mathbb{Z}$ is the constant sheaf. The cokernel is not a sheaf. To show this, consider the cover

$$U_1 = \mathbb{C} \setminus [0,\infty); \quad U_2 = \mathbb{C} \setminus (-\infty, 0]$$

and take $U = U_1 \cup U_2 = \mathbb{C} \setminus \{0\}$. Let $f(z) = z$, so $f \in \mathcal{O}_X^\star(U)$, but f is not in the image of $exp : \mathcal{O}_X(U) \to \mathcal{O}_X^\star(U)$ as there is no single-valued logarithm on $\mathbb{C} \setminus \{0\}$. Hence f defines a nonzero section of $(\operatorname{coker}\exp)(U)$. However, restricting to U_i, a single-valued branch of logarithm is defined, so f is in the image of $exp : \mathcal{O}_X(U_i) \to \mathcal{O}_X^\star(U_i)$. Thus $\operatorname{res}^U_{U_i} f = 1$, but $f \neq 1$, violating the first sheaf axiom.

Similarly, the image presheaf may not be a sheaf.

Definition. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define the *sheaf cokernel* and the *sheaf image* of φ to be the sheafifications of the presheaf cokernel and presheaf image respectively.

Remark. It turns out that the sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\star \longrightarrow 1$$

is an exact sequence of sheaves. In particular,

$$\ker \exp = 2\pi i\mathbb{Z}; \quad \operatorname{coker} \exp = 1$$

Remark. $\ker\varphi, \operatorname{coker}\varphi$ satisfy the category-theoretic definitions of kernels and cokernels. For kernels, the universal property to be satisfied is

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, dashed, "\exists!"'] \arrow[d, "\psi"] \arrow[dr, "0"] & \\
\ker\varphi \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G}
\end{tikzcd}$$

For cokernels, we reverse the arrows.

$$\begin{tikzcd}
& \mathcal{L} & \\
\operatorname{coker}\varphi \arrow[ur, dashed, "\exists!"] & \mathcal{F} \arrow[l] \arrow[u, "\psi"'] & \mathcal{G} \arrow[l, "\varphi"'] \arrow[ul, "0"'] \arrow[ll, bend left, "0"]
\end{tikzcd}$$

Definition. We say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$, written $\mathcal{F} \subseteq \mathcal{G}$, if there are inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with the restriction maps.

Kernels are examples of subsheaves.

2.6 Moving between spaces

Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y respectively.

Definition. The presheaf *pushforward* or *direct image* $f_\star\mathcal{F}$ is the presheaf on Y given by

$$f_\star\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

Proposition. The presheaf pushforward of a sheaf is a sheaf.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of U, and let $s \in f_\star\mathcal{F}(U)$ with $\operatorname{res}^U_{U_i} s = 0$ for each U_i. Then $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of $f^{-1}(U)$ and satisfies $\operatorname{res}^{f^{-1}(U)}_{f^{-1}(U_i)} s = 0$ in $\mathcal{F}(f^{-1}(U_i))$. So $s = 0$ as $\mathcal{F}$ is a sheaf.

Similarly, if $s_i \in f_\star\mathcal{F}(U)$ are compatible sections, then they can be glued into an element of $\mathcal{F}(f^{-1}(U))$. But this is precisely an element of $f_\star\mathcal{F}(U)$, as required. □

Definition. The *inverse image presheaf* $(f^{-1}\mathcal{G})^{\text{pre}}$ is the presheaf on X given by

$$(f^{-1}\mathcal{G})^{\text{pre}}(V) = {}^{\{(s_U, U) \mid f(V) \subseteq U, s_U \in \mathcal{G}(U)\}}\!\big/\!_{\sim}$$

where $\sim$ identifies pairs that agree on a smaller open set containing $f(V)$. The *inverse image sheaf* is $f^{-1}\mathcal{G} = ((f^{-1}\mathcal{G})^{\text{pre}})^{\text{sh}}$.

Example. The inverse image presheaf need not be a sheaf, even when f is an open map. Let Y be a topological space, and let $X = Y \sqcup Y$. Take $\mathcal{G} = \underline{\mathbb{Z}}$ the constant sheaf, and $\mathcal{F} = (f^{-1}\mathcal{G})^{\text{pre}}$. Let $U \subseteq Y$ be open, and let $V = f^{-1}(U)$. Then $\mathcal{F}(V) = \mathcal{G}(U) = \mathbb{Z}$, assuming U is connected. But $V = U \sqcup U$, so $\mathcal{F}^{\text{sh}}(V) = \mathcal{G}(U) \times \mathcal{G}(U) = \mathbb{Z}^2$.

Example. Let $\mathcal{F}$ be a sheaf on X, and let π be the map from X to a point. Then $f_\star\mathcal{F}$ is a sheaf on a point, which is just an abelian group, specifically $\mathcal{F}(\pi^{-1}(\{\bullet\})) = \mathcal{F}(X)$.

We will use the notation

$$\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$$

where Γ is called the *global sections*, and H_0 is called the *0th cohomology* with coefficients in $\mathcal{F}$.

For $p \in X$, $i : \{p\} \to X$. Let $\mathcal{G}$ be a sheaf on $\{p\}$, which is an abelian group A. Consider the sheaf $i_\star\mathcal{G}$ on X, defined by

$$(i_\star\mathcal{G})(U) = \begin{cases} 0 & \text{if } p \notin U \\ A & \text{if } p \in U \end{cases}$$

This is called the *skyscraper* at p with value A.

3 Schemes

We will now use the notation $f|_U$ for $\operatorname{res}^V_U f$.

3.1 Localisation

Definition. Let A be a ring and $S \subseteq A$ be a multiplicatively closed set. The *localisation* of A at S is

$$S^{-1}A = {}^{\{(a, s) \mid a \in A, s \in S\}}\!\big/\!_{\sim}$$

where

$$(a, s) \sim (a', s') \iff \exists s'' \in S,\ s''(as' - a's) = 0 \in A$$

Examples of multiplicatively closed sets include the set of powers of a fixed element, or the complement of a prime ideal. The pair (a, s) represents $\frac{a}{s}$. The extra s'' term represents a unit in this new ring, which may be needed in rings that are not integral domains.

Remark. The natural map $A \to S^{-1}A$ need not be injective, for example, if S contains a zero divisor.

3.2 Sheaves on a base

Definition. Let X be a topological space and $\mathcal{B}$ be a basis for the topology. A *sheaf on the base* $\mathcal{B}$ consists of assignments $B_i \mapsto F(B_i)$ of abelian groups, with restriction maps $\mathrm{res}^{B_i}_{B_j} : F(B_i) \to F(B_j)$ whenever $B_j \subseteq B_i$ such that,

(i) $\mathrm{res}^{B_i}_{B_i} = \mathrm{id}_{B_i}$;

(ii) $\mathrm{res}^{B_j}_{B_k} \circ \mathrm{res}^{B_i}_{B_j} = \mathrm{res}^{B_i}_{B_k}$

with the additional axioms that

(i) if $B = \bigcup B_i$ with $B, B_i \in \mathcal{B}$ and $f, g \in F(B)$ such that $f|_{B_i} = g|_{B_i}$ for all i, then $f = g$;

(ii) if $B = \bigcup B_i$ as above, with $f_i \in F(B_i)$ such that for all i, j and $B' \subseteq B_i \cap B_j$ with $B' \in \mathcal{B}$, $f_i|_{B'} = f_j|_{B'}$, then there exists $f \in F(B)$ with $f|_{B_i} = f_i$.

This is very similar to the definition of a sheaf, but only specified on the basis.

Proposition. Let F be a sheaf on a base $\mathcal{B}$ of X. This determines a sheaf $\mathcal{F}$ on X such that $\mathcal{F}(B) = F(B)$ for all $B \in \mathcal{B}$, agreeing with restriction maps. Moreover, $\mathcal{F}$ is unique up to unique isomorphism.

Proof. We first define the stalks using F:

$$\mathcal{F}_p = {\{(s_B, B) \mid p \in B \in \mathcal{B}, s_B \in F(B)\}}/{\sim}$$

We then use a sheafification idea to define $\mathcal{F}(U)$. The elements are the dependent functions $f \in \prod_{p \in U} \mathcal{F}_p$ such that for each $p \in U$, there exists a basic open set B containing p and a section $s \in F(B)$ such that $s_q = f_q$ in $\mathcal{F}_q$ for all $q \in B$. This is then clearly a sheaf. The natural maps $F(B) \to \mathcal{F}(B)$ are isomorphisms by the sheaf axioms. □

3.3 The structure sheaf

Recall that the distinguished opens U_f, U_g coincide if and only if f, g are powers of some $h \in A$. Also, if $U_f = U_g$ then $A_f \cong A_g$. Therefore, the assignment $U_f \mapsto A_f$ is well-defined.

Proposition. The assignment $U_f \mapsto A_f$ defines a sheaf of rings on the base $\{U_f\}$ of the topological space $\operatorname{Spec} A$.

Remark. If $\{U_{f_i}\}_{i \in I}$ covers $\operatorname{Spec} A$, there exists a finite subcover. Indeed, since the U_{f_i} cover $\operatorname{Spec} A$, there is no prime ideal $\mathfrak{p} \subseteq A$ containing all $(f_i)_{i \in I}$. Equivalently, $\sum_{i \in I}(f_i) = (1)$. In particular, $1 = \sum_{i \in J} a_i f_i$ for $J \subseteq I$ finite. So $\sum_{i \in J}(f_i) = (1)$, and thus $\{U_{f_i}\}_{i \in J}$ covers $\operatorname{Spec} A$. We say that $\operatorname{Spec} A$ is *quasi-compact*; traditionally the word 'compact' is reserved for Hausdorff spaces in the context of algebraic geometry.

Proof. We will check the axioms for the basic open set $B = \operatorname{Spec} A$; the general case follows by applying this result to a localisation. Suppose $\operatorname{Spec} A = \bigcup_{i=1}^n U_{f_i}$; this union is finite by the previous remark. Let $s \in A$ be such that $s|_{U_i} = 0$ for all i. By the definition of localisation, as the set $\{U_{f_i}\}$ is finite there exists m such that $f_i^m s = 0$ for all i. But note that $(1) = (f_i^m)_{i=1}^n$ for any $m > 0$ because the $\{U_{f_i}\}_{i=1}^n$ cover $\operatorname{Spec} A$. Thus $\{U_{f_i^m}\}_{i=1}^n$ cover $\operatorname{Spec} A$.

$$1 = \sum r_i f_i^m \implies s = \sum r_i f_i^m s = 0$$

Now suppose $\operatorname{Spec} A = \bigcup_{i \in I} U_{f_i}$, and $s_i \in A_{f_i}$ are elements that agree in $A_{f_i f_j}$. We need to build an element in A with these restrictions.

First, suppose I is finite. On U_{f_i}, we have chosen $\frac{a_i}{f_i^{\ell_i}} \in A_{f_i}$; we write $g_i = f_i^{\ell_i}$, noting that $U_{f_i} = U_{g_i}$. On the overlaps, by hypothesis we have

$$(g_i g_j)^{m_{ij}}(a_i g_j - a_j g_i) = 0$$

Rewriting this using the fact that $U_f = U_{f^k}$ for all $k > 0$, and assuming $m = m_{ij}$ by taking the largest, we obtain

$$b_i = a_i g_i^m; \quad h_i = g_i^{m+1}$$

so on each U_{h_i} we have chosen an element $\frac{b_i}{h_i}$. Now, as the $U_{h_i} = U_{f_i}$ cover $\operatorname{Spec} A$, we have $1 = \sum r_i h_i$ for some $r_i \in A$. We can thus construct $r = \sum r_i b_i$ with the r_i as above. This construction then has the correct restrictions to $\frac{b_i}{h_i}$ in U_{h_i}.

When I is infinite, choose $(f_i)_{i=1}^n$ such that the U_{f_i} for $i \in \{1, \dots, n\}$ form a cover, and use the finite case to build $r \in A$. This has the correct restrictions to the U_{f_i} for $i \in \{1, \dots, n\}$. Given $(f_1, \dots, f_n, f_\alpha) = A$, the same construction gives a new $r' \in A$, but then by the first sheaf axiom, $r = r'$. □

Definition. The *structure sheaf* on $\operatorname{Spec} A$ is the sheaf $\mathcal{O}_{\operatorname{Spec} A}$ associated to the sheaf on the base of distinguished opens mapping U_f to A_f.

Remark. The stalk $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ is equal to $A_{\mathfrak{p}}$.

3.4 Definitions and examples

Definition. A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings $\mathcal{O}_X$. An isomorphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism $\pi : X \to Y$ and an isomorphism $\mathcal{O}_Y \to \pi_\star \mathcal{O}_X$ of sheaves on Y.

Note that for $U \subseteq X$ open, U is naturally a ringed space with $\mathcal{O}_U(V) = \mathcal{O}_X(V)$.

Definition. An *affine scheme* is a ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Definition. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ where every point $p \in X$ has a neighbourhood U_p such that the ringed space $(U_p, \mathcal{O}_{U_p})$ is isomorphic to some affine scheme.

Proposition. Let X be a scheme, $U \subseteq X$ an open set, and $i : U \rightarrowtail X$ be the inclusion map. Then, the ringed space $(U, \mathcal{O}_U)$ is a scheme, where

$$\mathcal{O}_U = \mathcal{O}_X\Big|_U = i^{-1}\mathcal{O}_X$$

For example, take $X = \operatorname{Spec} A$ and $U = U_f$ for some $f \in A$. Then $(U, \mathcal{O}_U) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

Proof. Let $p \in U \subseteq X$. Since X is a scheme, we can find $\left(V_p, O_X|_{V_p}\right)$ inside X with $p \in V_p$, such that V_p is isomorphic to an affine scheme. Then take $V_p \cap U \subseteq U$ with structure sheaf given by the inclusion map. Note that $V_p \cap U$ may not be affine, but $V_p \cong \operatorname{Spec} B$, and the distinguished opens in $\operatorname{Spec} B$ form a basis. This reduces the problem to the example of a distinguished open set above. $\square$

Definition. *Affine space* of dimension n over k is defined to be

$$\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$$

Example. Let

$$U = \mathbb{A}^{n^2}_k \setminus \{\det(x_{ij}) = 0\}$$

which is the open set representing $GL_n(k)$. We will show that the multiplication map $U \times U \to U$ is a morphism of schemes.

Example. Let $U = \mathbb{A}^2_k \setminus (x, y)$. This is a scheme representing a plane without an origin. We claim that U is not an affine scheme. Suppose that U were affine; we aim to calculate $\mathcal{O}_U(U)$. Write

$$U_x = \mathbb{V}(x)^c \subseteq \mathbb{A}^2_k; \quad U_y = \mathbb{V}(y)^c \subseteq \mathbb{A}^2_k$$

These two open sets cover U, and

$$U_x \cap U_y = U_{xy} = \mathbb{A}^2_k \setminus \mathbb{V}(xy)$$

Then,

$$\mathcal{O}_U(U_x) = k[x, x^{-1}, y]; \quad \mathcal{O}_U(U_y) = k[x, y, y^{-1}]; \quad \mathcal{O}_U(U_x \cap U_y) = k[x, x^{-1}, y, y^{-1}]$$

The restriction maps $\mathcal{O}_U(U_x) \to \mathcal{O}_U(U_{xy})$ and $\mathcal{O}_U(U_y) \to \mathcal{O}_U(U_{xy})$ are the obvious ones. By the sheaf axioms,

$$\mathcal{O}_U(U) = k[x, x^{-1}, y] \cap k[x, y, y^{-1}] \subseteq k[x, x^{-1}, y, y^{-1}]$$

Thus, $\mathcal{O}_U(U) = k[x, y]$. This is a contradiction: one way to see this is that there exists a maximal ideal (x, y) in the ring of global sections in $(U, \mathcal{O}_U)$ with empty vanishing locus.

In general, if X is a scheme, $f \in \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$, and $p \in X$, then there is a well-defined stalk $\mathcal{O}_{X,p}$ at p, which is of the form $A_\mathfrak{p}$ up to isomorphism, where $\mathfrak{p}$ is a prime ideal. To say this, we are using an isomorphism of an open set V_p containing p to $\operatorname{Spec} A$. In particular, $A_\mathfrak{p}$ has a unique maximal ideal, namely $\mathfrak{p}A_\mathfrak{p}$. We say that f vanishes at p if its image in ${}^{A_\mathfrak{p}}\!/\!{}_{\mathfrak{p}A_\mathfrak{p}}$, or equivalently, $f \in \mathfrak{p}A_\mathfrak{p}$. As a consequence, the vanishing locus $\mathbb{V}(f) \subseteq X$ is well-defined.

3.5 Gluing sheaves

Let X be a topological space with a cover $\{U_\alpha\}$. Let $\{\mathcal{F}_\alpha\}$ be sheaves on $\{U_\alpha\}$, with isomorphisms

$$\varphi_{\alpha\beta} : \mathcal{F}_\alpha\big|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta\big|_{U_\alpha \cap U_\beta}$$

such that

$$\varphi_{\alpha\alpha} = \mathrm{id}; \quad \varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}; \quad \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$$

The last equation is called the *cocycle condition*. This combination of conditions resembles the definition of an equivalence relation, with reflexivity, symmetry, and transitivity.

We will construct a sheaf $\mathcal{F}$ on X. Given $V \subseteq X$ open, we define

$$\mathcal{F}(V) = \left\{ (s_\alpha) \in \prod_\alpha \mathcal{F}_\alpha(U_\alpha \cap V) \,\middle|\, \varphi_{\alpha\beta}\left(s_\alpha\big|_{V \cap U_\alpha \cap U_\beta} \right) = s_\beta\big|_{V \cap U_\alpha \cap U_\beta} \right\}$$

$\mathcal{F}$ is a presheaf. Indeed, given $(s_\alpha) \in \mathcal{F}(V)$ and $W \subseteq V$ open, we take

$$(s_\alpha)\big|_W = \left(\mathrm{res}^{V \cap U_\alpha}_{W \cap U_\alpha}(s_\alpha) \right)_\alpha$$

This lies in $\mathcal{F}(W)$ by the sheaf axioms. One check easily check that this is a sheaf.

Proposition. $\mathcal{F}|_{U_\gamma}$ and $\mathcal{F}_\gamma$ are canonically isomorphic as sheaves on U_γ.

Proof. First, we construct a map $\mathcal{F}_\gamma \to \mathcal{F}|_{U_\gamma}$. Let $V \subseteq U_\gamma$ and $s \in \mathcal{F}_\gamma(V)$. Define its image in $\mathcal{F}|_{U_\gamma}$ to be

$$\varphi_{\gamma\alpha}\left(s\big|_{V \cap U_\alpha} \right)_\alpha$$

We must check that this tuple lies in $\mathcal{F}|_{U_\gamma}(V) = \mathcal{F}(V)$.

$$\varphi_{\alpha\beta} \circ \varphi_{\gamma\alpha}\left(s\big|_{V \cap U_\alpha \cap U_\beta} \right) = \varphi_{\gamma\beta}\left(s\big|_{V \cap U_\alpha \cap U_\beta} \right)$$

□

3.6 Gluing schemes

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes with open sets $U \subseteq X, V \subseteq Y$, and let $\varphi : (U, \mathcal{O}_X|_U) \to (V, \mathcal{O}_Y|_V)$ be an isomorphism. The topological spaces X, Y can be glued on U, V using φ.

First, take $S = {}^{X \sqcup Y}\!/\!_{U \sim V}$. By definition of the quotient topology, the images of X and Y in S form an open cover, and their intersection is the image of U, or equivalently, the image of V. Now, we can glue the structure sheaves on these open sets as described in the previous subsection. Note that in this case, there is no cocycle condition.

Example (the bug-eyed line; the line with doubled origin). Let k be a field. Let $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[u]$. Let

$$U = \operatorname{Spec} k[t, t^{-1}] = \operatorname{Spec} k[t]_t = U_t \subseteq X; \quad V = \operatorname{Spec} k[u, u^{-1}] = \operatorname{Spec} k[u]_u = U_u \subseteq Y$$

We define the isomorphism $\varphi : U \to V$ given by $t \leftarrow u$. Technically, we define an isomorphism of rings $k[u, u^{-1}] \to k[t, t^{-1}]$ by $u \mapsto t$ and then apply Spec. At the level of topological spaces, $X = \mathbb{A}^1_k$ and $Y = \mathbb{A}^1_k$, so $U = \mathbb{A}^1_k \setminus \{(t)\}$ and $V = \mathbb{A}^1_k \setminus \{(u)\}$. Gluing along this isomorphism, we obtain a scheme S which is a copy of $\mathbb{A}^1_k$ but with two origins. Note that the generic points in X and Y lie in U and V respectively, and thus are glued into a single generic point in S.

Consider the open sets in S. Open sets entirely contained within X and Y yield open sets in S. We also have open sets of the form $W = S \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ where $\mathfrak{p}_i$ is contained in U or V. One example is $W = S$; we can calculate $\mathcal{O}_S(S)$ using the sheaf axioms, and one can show that it is isomorphic to $k[t]$. We can conclude that S is not an affine scheme, because there is a maximal ideal in $k[t]$ where the vanishing locus is precisely two points.

Example (the projective line). Let $X = \operatorname{Spec} k[t]$ and $Y = \operatorname{Spec} k[s]$, and define $U = \operatorname{Spec} k[t, t^{-1}]$, $V = \operatorname{Spec} k[s, s^{-1}]$ as above. We glue these schemes using the isomorphism $s \mapsto t^{-1}$, giving the projective line $\mathbb{P}^1_k$.

Proposition. $\mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k$.

Proof sketch. We use the same idea as in the previous example. The only elements of $k[t, t^{-1}]$ that are both polynomials in t and t^{-1} are the constants. □

In particular, $\mathbb{P}^1_k$ is not an affine scheme.

Example. We can similarly build a scheme S which is a copy of $\mathbb{A}^2_k$ with a doubled origin. This has the interesting property that there exist affine open subschemes $U_1, U_2 \subseteq S$ such that $U_1 \cap U_2$ is not affine; we can take U_1 and U_2 to be S but with one of the origins deleted. Note that $\mathbb{A}^1_k$ without the origin is affine.

Let $\{X_i\}_{i \in I}$ be schemes, $X_{ij} \subseteq X_i$ be open subschemes, and $f_{ij} : X_{ij} \to X_{ji}$ be isomorphisms such that

$$f_{ii} = \mathrm{id}_{X_i}; \quad f_{ij} = f_{ji}^{-1}; \quad f_{ik} = f_{jk} \circ f_{ij}$$

where the last equality holds whenever it is defined. Then there is a unique scheme X with an open cover by the X_i, glued along these isomorphisms. This is an elaboration of the above construction, which is discussed on the first example sheet.

Let A be a ring, and let $X_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$. Let $X_{ij} = \mathbb{V}\left(\frac{x_j}{x_i}\right)^c \subseteq X_i$. We define the isomorphisms $X_{ij} \to X_{ji}$ by $\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j}\left(\frac{x_i}{x_j}\right)^{-1}$. The resulting glued scheme is called *projective n-space*, denoted $\mathbb{P}^n_A$.

3.7 The Proj construction

Definition. A $\mathbb{Z}$-*grading* on a ring A is a decomposition

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

as abelian groups, such that $A_i A_j \subseteq A_{i+j}$.

Example. Let $A = k[x_0, \dots, x_n]$, and let A_d be the set of degree d homogeneous polynomials, together with the zero polynomial.

Example. Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal; that is, an ideal generated by homogeneous elements of possibly different degrees. Then, for $A = k[x_0, \dots, x_n]$, the ring ${}^{A}\!/\!{}_{I}$ is also naturally graded.

Note that by definition, A_0 is a subring of A. For simplicity, we will always assume in this course that the degree 1 elements of a graded ring generate A as an algebra over A_0. We also typically assume that $A_i = 0$ for $i < 0$. We define

$$A_+ = \bigoplus_{i \geq 1} A_i \subseteq A$$

This forms an ideal in A, called the *irrelevant ideal*. If A is a polynomial ring with the usual grading, the irrelevant ideal corresponds to the point $\mathbf{0}$ in the theory of varieties. This aligns with the definition of projective space in classical algebraic geometry, in which the point $\mathbf{0}$ is deleted.

A *homogeneous element* $f \in A$ is an element contained in some A_d. An ideal I of A is called *homogeneous* if it is generated by homogeneous elements.

Definition. Let A be a graded ring. $\operatorname{Proj} A$ is the set of homogeneous prime ideals in A that do not contain the irrelevant ideal. If $I \subseteq A$ is homogeneous, we define

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj} A \mid I \subseteq \mathfrak{p}\}$$

The *Zariski topology* on $\operatorname{Proj} A$ is the topology where the closed sets are of the form $\mathbb{V}(I)$ where I is a homogeneous ideal.

The Spec construction allows us to convert rings into schemes; the Proj construction allows us to convert graded rings into schemes. Unlike Spec, the construction of Proj is not functorial.

Let $f \in A_1$ and $U_f = \operatorname{Proj} A \setminus \mathbb{V}(f)$. Observe that the set $\{U_f\}_{f \in A_1}$ covers $\operatorname{Proj} A$, because the f generate the unit ideal. The ring $A\left[\frac{1}{f}\right] = A_f$ is naturally $\mathbb{Z}$-graded by defining $\deg \frac{1}{f} = -\deg f$. Note that A_f may have negatively graded elements, even though A does not.

Example. Let $A = k[x_0, x_1]$ and $f = x_0$. Then in $A\left[\frac{1}{f}\right] = k[x_0, x_1, x_0^{-1}]$, the degree zero elements include k and elements such as $\frac{x_1}{x_0}, \frac{x_1^2 + x_1 x_0}{x_0^2}$. There are degree one elements such as $\frac{x_1^2}{x_0}$.

Proposition. There is a natural bijection

$$\{\text{homogeneous prime ideals in } A \text{ that miss } f\} \leftrightarrow \left\{\text{prime ideals in } (A_f)_0\right\}$$

Note also that the set of homogeneous prime ideals in A that miss f are naturally in bijection with the homogeneous prime ideals in A_f.

Proof. Suppose $\mathfrak{q}$ is a prime ideal in $\left(A\left[\frac{1}{f}\right]\right)_0$. Then let $\psi(\mathfrak{q})$ be the ideal

$$\psi(\mathfrak{q}) = \left(\bigcup_{d\geq 0}\left\{a \in A_d \,\middle|\, \frac{a}{f^d} \in \mathfrak{q}\right\} \subseteq A\right)$$

One can check that this is prime. Now suppose $\mathfrak{p}$ is a homogeneous prime ideal missing f. Define $\varphi(\mathfrak{p})$ to be

$$\varphi(\mathfrak{p}) = \left(p \cdot A\left[\frac{1}{f}\right] \cap \left(A\left[\frac{1}{f}\right]\right)_0\right)$$

This ideal is also prime.

One can easily check that $\varphi \circ \psi$ is the identity. For the other direction, suppose $\mathfrak{p}$ is a homogeneous prime ideal missing f; we show that $\mathfrak{p} = \psi(\varphi(\mathfrak{p}))$ by antisymmetry. If $a \in \mathfrak{p} \in A_d$, then $\frac{a}{f^d} \in \varphi(\mathfrak{p})$, so $a \in \psi(\varphi(\mathfrak{p}))$ by construction. Conversely, if $a \in \psi(\varphi(\mathfrak{p}))$, then $\frac{a}{f^d} \in \varphi(\mathfrak{p})$ for some d, so there exists $b \in \mathfrak{p}$ such that $\frac{b}{f^e} = \frac{a}{f^d}$ in $A\left[\frac{1}{f}\right]$. Hence for some $k \geq 0$, we have $f^k(f^d b - f^e a) = 0$, and $f^{e+k} \notin \mathfrak{p}$. But by primality, $a \in \mathfrak{p}$, as required. □

The bijection constructed is compatible with ideal containment, so is a homeomorphism of topological spaces

$$U_f \leftrightarrow \operatorname{Spec}(A_f)_0$$

Thus $\operatorname{Proj} A$ is covered by open sets homeomorphic to an affine scheme. If $f, g \in A_1$, then $U_f \cap U_g$ is naturally homeomorphic to

$$\left(\operatorname{Spec} A\left[\frac{1}{f}\right]\right)_0\left[\frac{f}{g}\right] = \operatorname{Spec}\left(A\left[f^{-1}, g^{-1}\right]\right)_0$$

Take the open cover $\{U_f\}$ with structure sheaf $\mathcal{O}_{\operatorname{Spec}(A_f)_0}$ on each U_f, and isomorphisms on $U_f \cap U_g$ by the condition above. The cocycle condition follows from the formal properties of the localisation. Therefore, $\operatorname{Proj} A$ is a scheme.

If $A = k[x_0, \dots, x_n]$ with the standard grading, we write $\mathbb{P}^n_k$ for $\operatorname{Proj} A$.

4 Morphisms

4.1 Morphisms of ringed spaces

Let $(X, \mathcal{O}_X)$ be a scheme. The stalks $\mathcal{O}_{X,\mathfrak{p}}$ are local rings: they have a unique maximal ideal, which is the set of all non-unit elements. Given $f \in \mathcal{O}_X(U)$, we can meaningfully ask whether f vanishes at $\mathfrak{p}$; that is, if the image of f in $\mathcal{O}_{X,\mathfrak{p}}$ is contained in the maximal ideal.

Definition. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous function $f : X \to Y$ and a morphism $f^\sharp : \mathcal{O}_Y \to f_\star \mathcal{O}_X$ between sheaves of rings on Y.

$f^\sharp$ represents function composition with f^{-1}, although the ring $\mathcal{O}_X$ may not be a ring of functions. It is possible to find a morphism $(f, f^\sharp)$ between schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ such that there exists $q \in U \subseteq Y$ and $h \in \mathcal{O}_Y(U)$ such that h vanishes at q but $f^\sharp(h) \in \mathcal{O}_X(f^{-1}(U))$ does not vanish at some $p \in X$ with $f(p) = q$. This motivates the definition of a morphism of schemes.

Let $f : X \to Y$ be a morphism of ringed spaces. Given any point $p \in X$, there is an induced map $f^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$. Explicitly, given $s \in \mathcal{O}_{Y,f(p)}$, we can represent it by (s_U, U) where U is open, $f(p) \in U$, and $s_U \in \mathcal{O}_Y(U)$. Now, $f^\sharp(s_U) \in \mathcal{O}_X(f^{-1}(U))$, so the pair $(f^\sharp(s_U), f^{-1}(U))$ defines an element of $\mathcal{O}_{X,p}$.

Definition. A ringed space $(X, \mathcal{O}_X)$ is called a *locally ringed space* if for all $p \in X$, the stalk $\mathcal{O}_{X,p}$ is is a local ring. A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that if $\mathfrak{m}_p$ denotes the maximal ideal in $\mathcal{O}_{X,p}$, then $f^\sharp(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p$.

This encapsulates the idea that functions vanishing on the codomain must also vanish on the domain after the inverse image, as the maximal ideal represents functions vanishing at the point.

4.2 Morphisms of schemes

Note that all schemes are locally ringed spaces.

Definition. A *morphism of schemes* $X \to Y$ is a morphism of locally ringed spaces $X \to Y$.

Theorem. There is a natural bijection

$$\{\text{morphisms of schemes } \operatorname{Spec} B \to \operatorname{Spec} A\} \leftrightarrow \{\text{homomorphisms of rings } A \to B\}$$

Proof. First, recall that a section s of a sheaf $\mathcal{F}$ on U is a coherent collection of elements of the stalks $s(p) \in \mathcal{F}_p$ for all $p \in U$. We will construct a map of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ for every ring homomorphism $A \to B$, and then show that every morphism of schemes arises in this way.

Let $\varphi : A \to B$ be a ring homomorphism. Let $\varphi^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ be the map of topological spaces; this is a continuous function. We now build

$$\varphi^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to \varphi^{-1}_\star \mathcal{O}_{\operatorname{Spec} B}$$

At the level of stalks, the map $A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ is induced by φ by mapping $\frac{a}{s}$ to $\frac{\varphi(a)}{\varphi(s)}$. This is well-defined, as for $s \notin \varphi^{-1}(\mathfrak{p})$, then $\varphi(s) \notin \mathfrak{p}$. Observe that this is automatically a local homomorphism.

We must now show that this choice of maps on stalks extends to a map between sheaves. Given $U \subseteq \operatorname{Spec} A$, we need to define

$$\varphi^\sharp : \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} B}((\varphi^{-1})^{-1}(U))$$

An element $s \in \mathcal{O}_{\operatorname{Spec} A}(U)$ is a collection of assignments $(\mathfrak{p} \mapsto s(\mathfrak{p}))_{\mathfrak{p}\in U}$ for $\mathfrak{p} \in U$ and $s(\mathfrak{p}) \in A_{\mathfrak{p}}$. We then define $\varphi^\sharp$ by

$$(\mathfrak{p} \mapsto s(\mathfrak{p}))_{\mathfrak{p}\in U} \mapsto (\mathfrak{q} \mapsto \varphi_{\mathfrak{q}}(s(\varphi^{-1}(\mathfrak{q}))))_{\mathfrak{q}\in(\varphi^{-1})^{-1}(U)}$$

One can check that the gluing conditions are satisfied.

Conversely, suppose $(f, f^\sharp) : \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of schemes. Using the fact that we have a map of global sections $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B)$, we obtain a ring homomorphism $g : A \to B$. We must check that $g^{-1} : \operatorname{Spec} B \to \operatorname{Spec} A$ gives the correct map f on topological spaces, and that the construction above yields the correct map $f^\sharp$ on sheaves. The maps on stalks are compatible with restriction, so the following diagram commutes for all $\mathfrak{p} \in \operatorname{Spec} B$.

$$\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\
\mathcal{O}_{\operatorname{Spec} A, f(\mathfrak{p})} \arrow[r] & \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}
\end{tikzcd}$$

Equivalently, the following diagram commutes for all $\mathfrak{p} \in \operatorname{Spec} B$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A_{f(\mathfrak{p})} \arrow[r] & B_{\mathfrak{p}}
\end{tikzcd}$$

Since the morphism is local, $(f^\sharp)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p})A_{f(\mathfrak{p})}$. As the above diagram commutes, $g^{-1} = f$ as maps of topological spaces, and the maps of structure sheaves agree at the level of stalks by construction so they must agree everywhere. □

4.3 Immersions

Definition. Let X, Y be schemes. A morphism of schemes $f : X \to Y$ is an *open immersion* if f induces an isomorphism of X onto an open subscheme $\left(U, \mathcal{O}_Y|_U\right)$ of Y. A morphism $f : X \to Y$ is a *closed immersion* if f is a homeomorphism onto a closed subset of Y, and $g^\sharp : \mathcal{O}_Y \to g_\star \mathcal{O}_X$ is surjective.

Example. Let $k[t] \to {}^{k[t]}\!/_{(t^2)}$. The induced map $\operatorname{Spec} {}^{k[t]}\!/_{(t^2)} \to \operatorname{Spec} k[t]$ is a closed immersion. More generally, let A be a ring and I be an ideal in A. Then the induced map $\operatorname{Spec} {}^{A}\!/_{I} \to \operatorname{Spec} A$ is a closed immersion.

Definition. Let Y be a scheme. A *closed subscheme* of Y is an equivalence class of closed immersions $X \to Y$, where we say $f : X \to Y$ and $f' : X' \to Y$ are equivalent if there is a commutative triangle

$$\begin{tikzcd}
X \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& Y &
\end{tikzcd}$$

4.4 Fibre products

The notion of fibre product will simultaneously generalise the notions of product, intersections of closed subschemes, and inverse images of subschemes (such as points) along morphisms.

Definition. Consider a diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

The *fibre product* is a scheme $X \times_S Y$ making the following diagram commute:

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

such that for any other scheme Z together with morphisms q_X, q_Y completing the square, there is a unique factorisation through $X \times_S Y$, making the following diagram commute.

$$\begin{tikzcd} Z \arrow[dr, dashed] \arrow[drr, bend left, "q_X"] \arrow[ddr, bend right, "q_Y"'] & & \\ & X \times_S Y \arrow[r, "p_X"'] \arrow[d, "p_Y"] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

Note that as this is a definition by universal property, if $X \times_S Y$ exists, it is unique up to unique isomorphism. The fibre product is schemes is the category-theoretic *pullback*.

Example. (i) In the category of sets, the fibre product of the diagram

$$\begin{tikzcd} & X \arrow[d, "r_X"] \\ Y \arrow[r, "r_Y"'] & S \end{tikzcd}$$

is the set

$$X \times_S Y = \{(x, y) \in X \times Y \mid r_X(x) = r_Y(y)\}$$

(ii) In the category of topological spaces, the fibre product is defined to be the same set, assigning $X \times_S Y$ the subspace topology as a subset of $X \times Y$.

(iii) Let $r_X : X \to S$ be a map of sets, and let $Y = \{\star\}$ with $r_Y(\star) = s \in S$. Then

$$X \times_S Y = r_X^{-1}(s)$$

(iv) Let $r_X : X \to S$ and $r_Y : Y \to S$ be inclusions of subsets. Then

$$X \times_S Y = X \cap Y$$

Theorem. Fibre products of schemes exist.

Proof sketch. Step 1. Let X, Y, S be affine schemes, with associated rings A, B, R. Then the fibre product $X \times_S Y$ exists, and is isomorphic to $\operatorname{Spec}(A \otimes_R B)$. Note that the tensor product is the category-theoretic pushout in the category of rings. We must now check that the universal property of the fibre product is satisfied. Consider the commutative square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

If Z is an affine scheme, the result holds. It is a general fact that a map of schemes $Z \to \operatorname{Spec}(A \otimes_R B)$ is the same data as a map $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$.

Step 2. Let X, Y, S be arbitrary schemes. If $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme, then $U \times_S Y$ also exists, by taking the inverse image of U under the projection $X \times_S Y \to X$ endowed with the structure of an open subscheme.

Step 3. If X is covered by open subschemes $\{X_i\}$, then if $X_i \times_S Y$ exists for all i, then $X \times_S Y$ exists, by gluing each of the $X_i \times_S Y$ together. Note that the ability to glue these schemes together relies on Step 2, and the fact that there is no cocycle condition.

Step 4. If Y and S are affine, then $X \times_S Y$ exists by Step 3, by covering X by affine subschemes. As X and Y are interchangeable, $X \times_S Y$ exists for any X and Y as long as S is affine.

Step 5. Now, cover S by affine subschemes $\{S_i\}$. Let X_i, Y_i be the preimages of of S_i in X and Y respectively. Now, $X_i \times_{S_i} Y_i$ exists. Observe by the universal property that $X_i \times_{S_i} Y_i = X_i \times SY_i$. Finally, gluing gives $X \times_S Y$ as required. $\square$

Example. (i) We have
$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}$$
where the map $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$ is induced by the ring homomorphism $\mathbb{Z} \to \mathbb{C}$, and the map $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is induced locally by the inclusion $\mathbb{Z} \to \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$. Note also that
$$\mathbb{Z}[\mathbf{x}] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[\mathbf{x}]$$

(ii) Let $C = \operatorname{Spec} \mathbb{C}[x,y]/(y - x^2)$ and $L = \operatorname{Spec} \mathbb{C}[x,y]/(y)$. We have natural closed immersions $C \to \mathbb{A}^2_{\mathbb{C}}$ and $L \to \mathbb{A}^2_{\mathbb{C}}$. One can show that
$$C \times_{\mathbb{A}^2_{\mathbb{C}}} L = \operatorname{Spec} \mathbb{C}[x]/(x^2)$$
representing the intersection.

4.5 Schemes over a base

In scheme theory, we often fix a scheme S called the *base scheme*, and consider other schemes with a fixed map to S. These form a category of schemes *over* S, where the morphisms are the morphisms of schemes $f : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

This is known as Grothendieck's *relative point of view*. Typically, S is the spectrum of a field or a ring. Note that every scheme has a unique morphism to $\operatorname{Spec} \mathbb{Z}$, so the category of schemes is isomorphic to the category of schemes over $\operatorname{Spec} \mathbb{Z}$. The product of X and Y in the category of schemes over S is the fibre product $X \times_S Y$. Analogously, in commutative algebra, we often consider algebras of a fixed ring, and the category of rings is isomorphic to the category of $\mathbb{Z}$-algebras.

4.6 Separatedness

Recall that a topological space X is Hausdorff if and only if the diagonal $\Delta_X \subseteq X \times X$ is closed.

Definition. Let $X \to S$ be a morphism of schemes. Then the *diagonal* is the morphism $\Delta_{X/S} : X \to X \times_S X$ induced using the universal property by the following diagram.

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

We write Δ for $\Delta_{X/S}$ if X and S are clear from context.

Remark. If U, V are open subschemes of X and $S = \operatorname{Spec} k$ for a field k, then

$$\Delta^{-1}(U \times_S V) = U \cap V$$

Definition. A morphism $X \to S$ is *separated* if $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion.

Example. Let $X = \operatorname{Spec} \mathbb{C}[t]$, let $S = \operatorname{Spec} \mathbb{C}$, and induce the map $X \to S$ by the $\mathbb{C}$-algebra homomorphism $\mathbb{C} \to \mathbb{C}[t]$. Then

$$X \times_S X = \operatorname{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t])$$

and the diagonal map Δ is induced by the multiplication map

$$\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \to \mathbb{C}[t]$$

Note that Δ is closed, as the map $\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] \to \mathbb{C}[t]$ is surjective.

Proposition. Let $g : X \to S$ be a morphism of schemes. Then there is a factorisation of $\Delta_{X/S}$ as follows.

$$\begin{tikzcd} & U \arrow[dr, "\text{open immersion}"] & \\ X \arrow[ur, "\text{closed immersion}"] \arrow[rr, "\Delta_{X/S}"'] & & X \times_S X \end{tikzcd}$$

We say that $g : X \to S$ is a *locally closed immersion*.

Proof. Let S be covered by open affine subschemes $\{V_i\}$, and suppose X is covered by open affine subschemes $\{U_{ij}\}$, where for some fixed i, the U_{ij} cover $g^{-1}(V_i)$. We have morphisms $U_{ij} \to V_i$ induced by

$$\begin{tikzcd} U_{ij} \arrow[r] & g^{-1}(V_i) \arrow[r] \arrow[d] & V_i \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

where the commutative square is a fibre product. Observe that $U_{ij} \times_{V_i} U_{ij}$ is affine and open in $X \times_S X$, and their union contains the image of the diagonal $\Delta_{X/S}$. Also,

$$\Delta^{-1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij} \subseteq X$$

Let U be the union of the $U_{ij} \times_{V_i} U_{ij}$ over all i, j. Then the second map in the statement is clearly an open immersion. Observe that to check if $f : T \to T'$ is a closed immersion, it suffices to check locally on the codomain. For each U_{ij}, the diagonal is a map $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$, which one can show is a closed immersion. $\square$

Proposition. If $X \to S$ is a morphism of affine schemes, then $\Delta_{X/S}$ is a closed immersion.

Proof. Let $X = \operatorname{Spec} A$, $S = \operatorname{Spec} B$, and let the map $X \to S$ be given by a map $B \to A$. Then the map $A \otimes_B A \to A$ is surjective as required. $\square$

Thus every morphism of affine schemes is separated.

Corollary. Let $X \to S$ be a morphism of schemes. If the image of $\Delta_{X/S}$ is closed as a topological subspace, then $X \to S$ is separated.

Proof. A locally closed immersion onto a closed subset is a closed immersion. $\square$

Example. (i) Recall the bug-eyed line

$$\mathbb{A}^1_k \sqcup \mathbb{A}^1_k \big/ \sim$$

where if $U = \mathbb{A}^1_k \setminus \{0\} \subseteq \mathbb{A}^1_k$ and V is defined similarly, we define the isomorphism $V \to U$ by the map $u \mapsto t : k[u, u^{-1}] \to k[t, t^{-1}]$. We claim that the bug-eyed line is not separated over $\operatorname{Spec} k$. We can compute $X \times_S X$ by the gluing construction of the fibre product. This is a plane with doubled axes and four origins. The diagonal only contains two of the four origins, and this is not a closed subset.

(ii) Open and closed immersions are are always separated.

(iii) All monomorphisms are separated.

(iv) Compositions of separated morphisms are separated.

(v) Suppose $X \to S$ is separated and $S' \to S$ is an embedding. Then the map $X \times_S S' \to S'$ that comes from

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is also separated. This is called a *base extension*: the right-hand side of the diagram is the original morphism $X \to S$, and the left-hand side can be thought of as the same morphism under a base change.

Proposition. Let R be a ring. The morphism $\mathbb{P}^n_R \to \operatorname{Spec} R$ is separated.

Proposition. We want to show that the map Δ in the following diagram is closed, where the commutative square is a fibre product.

$$\begin{tikzcd} \mathbb{P}^n_R \arrow[r, "\Delta"] & \mathbb{P}^n_R \times_R \mathbb{P}^n_R \arrow[r] \arrow[d] & \mathbb{P}^n_R \arrow[d] \\ & \mathbb{P}^n_R \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

It suffices to check this result on an open cover of $\mathbb{P}^n_R \times_R \mathbb{P}^n_R$. Let $A = R[x_0, \dots, x_n]$ with the usual grading, so $\operatorname{Proj} A = \mathbb{P}^n_R$. Then let $U_i = \operatorname{Spec}\left(A\left[\frac{1}{x_i}\right]\right)_0$. These U_i form an open cover of $\mathbb{P}^n_R$. Now,

$$U_i \times_R U_j = \operatorname{Spec} R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \dots, \frac{y_n}{y_j}\right]$$

Observe that the restriction of Δ to $\Delta^{-1}(U_i \times_R U_j)$ is

$$U_i \cap U_j \to U_i \times_R U_j$$

given on rings by the map

$$R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_j}\right]\left[\frac{x_i}{x_j}\right] \leftarrow R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \dots, \frac{y_n}{y_j}\right]$$

by changing y_k into x_k. This is surjective, and the $U_i \times_R U_j$ cover $\mathbb{P}^n_R \times_R \mathbb{P}^n_R$, so Δ is closed.

Definition. Let $k = \overline{k}$ be an algebraically closed field. Let $X \to \operatorname{Spec} k$ be a scheme over $\operatorname{Spec} k$. We say that X is of *finite type* over $\operatorname{Spec} k$ if there is a cover of X by affines $\{U_\alpha\}_\alpha$ such that $\mathcal{O}_X(U_\alpha)$ is finitely generated k-algebra. We say that X is *reduced* if for all open $U \subseteq X$, $\mathcal{O}_X(U)$ has no nilpotent elements.

Definition. A morphism $X \to \operatorname{Spec} k$ is a *variety* if it is reduced, of finite type, and separated.

4.7 Properness

Definition. Let $f : X \to S$ be a morphism. Then f is of *finite type* if there exists an affine cover of S by open $\{V_\alpha\}_\alpha$ where $V_\alpha = \operatorname{Spec} A_\alpha$, and covers $\{U_{\alpha\beta}\}_\beta$ of $f^{-1}(V_\alpha)$ by open affine subschemes with $U_{\alpha\beta} = \operatorname{Spec} B_{\alpha\beta}$, such that $B_{\alpha\beta}$ is a finitely generated A_α-algebra, and $\{U_\alpha\beta\}_\beta$ can be chosen to be finite.

Definition. A moprhism $f : X \to S$ is *closed* if it is closed as a map of topological spaces. It is *universally closed* if for any $S' \to S$, the induced map $X \times_S S' \to S'$ is also closed. f is *proper* if it is separated, of finite type, and universally closed.

Example. (i) Closed immersions are proper.

(ii) The obvious map $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper, because it is not universally closed. Indeed, consider the fibre product

$$\begin{tikzcd} \mathbb{A}^2_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\ \mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Consider $Z \subseteq \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ given by the vanishing locus of $xy - 1$. Then the projection of Z onto each axis is not Zariski closed.

(iii) The bug-eyed line is neither separated nor universally closed.

Remark. If $X \to S$ is universally closed, then any base extension $X \times_S S' \to S'$ is also universally closed. Similarly, separatedness, properness and being of finite type are stable under base extension.

Proposition. Let R be a commutative ring. Then the morphism $\mathbb{P}^n_R \to \operatorname{Spec} R$ is proper.

Proof. We have already shown that $\mathbb{P}^n_R \to \operatorname{Spec} R$ is separated. It is of finite type by construction. It suffices to prove that the morphism is universally closed for $R = \mathbb{Z}$, because $\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R$. We must show that for any $Y \to \operatorname{Spec} \mathbb{Z}$, the base extension $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y \to Y$ is closed. But Y is covered by affine schemes of the form $\operatorname{Spec} R$, and closedness is local on the codomain, it suffices to show that $\mathbb{P}^n_R \to \operatorname{Spec} R$ is closed.

Let $Z \subseteq \mathbb{P}^n_R$ be Zariski closed, so Z is the vanishing locus of homogeneous polynomials $\{g_1, g_2, \dots\}$. We want to show that if π is the map $\mathbb{P}^n_R \to \operatorname{Spec} R$, then $\pi(Z)$ is closed. We need to find equations for $\pi(Z)$, or equivalently, we need to characterise the prime ideals $\mathfrak{p}$ of R such that $\pi^{-1}(\mathfrak{p}) \cap Z$ is nonempty. Let $k(\mathfrak{p}) = FF\left({}^{R}\!/_{\mathfrak{p}}\right)$. We have a morphism $\operatorname{Spec} k(\mathfrak{p}) \to \operatorname{Spec} R$. Let $Z_{\mathfrak{p}} = Z \times_{\operatorname{Spec} R} \operatorname{Spec} k(\mathfrak{p})$; we want to know for which $\mathfrak{p}$ this scheme is nonempty. If we take the equations $g_1, g_2, \dots$ and reduce modulo $\mathfrak{p}$, we obtain equations $\overline{g}_1, \overline{g}_2, \dots$ which are homogeneous polynomials in $k(\mathfrak{p})$. Thus $Z_{\mathfrak{p}}$ is nonempty if and only if $\overline{g}_1, \overline{g}_2, \dots$ cut out more than the origin in $\mathbb{A}^{n+1}_{k(\mathfrak{p})}$. In particular, $Z_{\mathfrak{p}}$ is nonempty if and only if

$$\sqrt{(\overline{g}_1, \overline{g}_2, \dots)} \not\supseteq (x_0, \dots, x_n); \quad \mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$$

Equivalently, for all positive integers d,

$$(x_0, \dots, x_n)^d \not\subseteq (\overline{g}_1, \overline{g}_2, \dots)$$

Write $A = R[\mathbf{x}]$ with the usual grading. The non-containment condition above holds if and only if the map

$$\bigoplus_i A_{d - \deg g_i} \to A_d$$

given by $f_i \mapsto f_i g_i$ in the ith factor is not surjective modulo $\mathfrak{p}$, or equivalently in $k(\mathfrak{p})$, for all degrees d. This condition is given by the maximal minors of the matrix associated to $\bigoplus_i A_{d-\deg g_i} \to A_d$, which is a set of infinitely many polynomials, each in the coefficients of the g_i. □

4.8 Valuative criteria

From here, we will assume that all schemes are Noetherian; that is, it has a finite cover by spectra of Noetherian rings.

Definition. A *discrete valuation ring* is a local principal ideal domain.

Example. (i) $\mathbb{C}[\![t]\!]$ is a discrete valuation ring.

(ii) $\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \,\middle|\, g(0) \neq 0 \right\}$ is a discrete valuation ring.

(iii) Similarly, $\mathbb{Z}_{(p)}, \mathbb{Z}_p$ are discrete valuation rings, where $\mathbb{Z}_{(p)}$ denotes the localisation of $\mathbb{Z}$ at the prime ideal (p), and $\mathbb{Z}_p$ denotes the p-adic integers.

We will often drop the word 'discrete'.

Remark. Let A be a valuation ring. In discrete valuation rings, every nonzero prime ideal is maximal, so $\operatorname{Spec} A$ consists of two points, (0) and the unique maximal ideal $\mathfrak{m}$. The topology on $\operatorname{Spec} A = \{(0), \mathfrak{m}\}$ has the property that (0) is dense and $\mathfrak{m}$ is closed. This is called the *Sierpiński topology*.

Any generator π for $\mathfrak{m}$ is called a *uniformiser* or a *uniformising parameter*. For example, in $\mathbb{C}[\![t]\!]$, every power series with nonzero constant term is a unit, and t is a uniformiser.

Given a uniformiser, any nonzero element $a \in A$ can be written as $u\pi^k$ where u is a unit and k is a unique natural number called the *valuation* of a. This gives a map $A \setminus \{0\} \to \mathbb{N}$ mapping a value a to its valuation; this is independent of the choice of uniformiser.

The field of fractions of A is a *valued field* $K = FF(A)$; the valuation extends to a multiplicative function $K \setminus \{0\} \to \mathbb{Z}$ given by the difference of valuations of the numerator and denominator.

Example. Let $A = k[\![t]\!]$, then $K = k(\!(t)\!)$ is the field of Laurent series in one variable in k. The valuation is the order of vanishing at zero.

One can consider the open immersion $\operatorname{Spec} K \to \operatorname{Spec} A$ as the inclusion from a disc with a punctured origin to a disc.

Theorem. Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if for any (discrete) valuation ring A with function field K and diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

then there exists at most one lift $\operatorname{Spec} A \to X$ that makes the following diagram commute.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Similarly, f is universally closed if and only if there exists at least one lift $\operatorname{Spec} A \to X$ that makes the diagram commute.

In particular, a morphism is proper if and only if there is a unique lift, and the morphism is of finite type. The proof is omitted.

Remark. (i) The map $\mathbb{P}^n_R \to \operatorname{Spec} R$ is proper.

(ii) The map $\mathbb{A}^n_R \to \operatorname{Spec} R$ is not proper, but is separated.

(iii) Closed immersions are proper. In particular, if $Z \to \mathbb{P}^n_R$ is closed, then $Z \to \operatorname{Spec} R$ is proper.

(iv) Compositions of proper (respectively separated) morphisms are proper (separated).

(v) If $f : X \to Y$ is proper, then for any $Y' \to Y$, the base extension $X \times_Y Y' \to Y'$ is also proper.

Example. We show that $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper by showing it is not universally closed. Write $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, and consider $A = k[\![t]\!]$ and $K = k(\!(t)\!)$.

$$\begin{tikzcd} \operatorname{Spec} k(\!(t)\!) \arrow[r, "\varphi"] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\ \operatorname{Spec} k[\![t]\!] \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

The map $\operatorname{Spec} k[\![t]\!] \to \operatorname{Spec} k$ is the obvious morphism. Let φ be induced by the map on rings $k[x] \to k(\!(t)\!)$ given by $x \mapsto \frac{1}{t}$. Then the map does not factor through $\operatorname{Spec} k[\![t]\!] \to \operatorname{Spec} k(\!(t)\!)$, as required. However, if we replace $\mathbb{A}^1_k$ with $\mathbb{P}^1_k$, there is always an affine chart in $\mathbb{P}^1$ such that φ is of the form $x \mapsto t$.

5 Modules over the structure sheaf

5.1 Definitions

Example. Let $\mathbb{C}P^n$ be the variety $\mathbb{C}^{n+1} \setminus \{0\}$ modulo scaling by $\mathbb{C}$. We have a structure sheaf $\mathcal{O}_{\mathbb{C}P^n}$, where if $U \subseteq \mathbb{C}P^n$ is Zariski open, we define

$$\mathcal{O}_{\mathbb{C}P^n}(U) = \left\{ \frac{P(\mathbf{x})}{Q(\mathbf{x})} \,\middle|\, P, Q \text{ homogeneous of the same degree, and the ratio is regular at all } p \in U \right\}$$

For any integer d, we can consider a sheaf $\mathcal{O}_{\mathbb{C}P^n}(d)$ given by

$$\mathcal{O}_{\mathbb{C}P^n}(d)(U) = \left\{ \frac{P(\mathbf{x})}{Q(\mathbf{x})} \,\middle|\, P, Q \text{ homogeneous, } \deg P - \deg Q = d, \text{ and regular at all } p \in U \right\}$$

This is a sheaf of groups, but not a sheaf of rings as it is not closed under multiplication for $d \neq 0$. Note that $\mathcal{O}_{\mathbb{C}P^n}(d)(U)$ is a module over $\mathcal{O}_{\mathbb{C}P^n}(U)$, and the multiplication commutes with restriction.

Example. Let A be a ring, and let M be an A-module. We define the sheaf $\mathcal{F}_M = M^{\mathrm{sh}}$ on $\operatorname{Spec} A$ as follows. If $U \subseteq \operatorname{Spec} A$ is a distinguished open $U = U_f$, then we set

$$\mathcal{F}_M(U) = M_f$$

which is the module M localised at f. This defines a sheaf on a base, and hence extends to a unique sheaf on $\operatorname{Spec} A$.

Definition. Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-modules* on X is a sheaf $\mathcal{F}$ of abelian groups together with a multiplication $\mathcal{F}(U) \times \mathcal{O}_X(U) \to \mathcal{F}(U)$ that makes $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module, that is compatible with restriction.

$$\begin{tikzcd}
\mathcal{F}(V) \times \mathcal{O}_X(V) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[d] \\
\mathcal{F}(U) \times \mathcal{O}_X(U) \arrow[r] & \mathcal{F}(U)
\end{tikzcd}$$

Similarly, we can define a sheaf of $\mathcal{O}_X$-algebras. A morphism between sheaves of modules $\varphi : \mathcal{F} \to \mathcal{G}$ on X is a homomorphism of sheaves of abelian groups that is compatible with multiplication.

Given morphisms of sheaves of modules on X, we can locally take kernels, cokernels, images, direct sums, tensor products, hom functors, and all of these extend to sheaves of modules. In the case of cokernels, images, and tensor products, we require a sheafification step. For example, the presheaf tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ associated to an open set $U \subseteq X$ is given by $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$; the sheaf tensor product is given by sheafification.

Given a morphism of ringed spaces or schemes $f : X \to Y$, the pushforward of an $\mathcal{O}_X$-module $\mathcal{F}$ is the sheaf of abelian groups $f_\star\mathcal{F}$. As a morphism of ringed spaces, we also have a map $f^\sharp : \mathcal{O}_Y \to f_\star\mathcal{O}_X$, giving $f_\star\mathcal{F}$ an $\mathcal{O}_Y$-module structure. Given an open set $U \subseteq Y$, $a \in \mathcal{O}_Y(U)$, and $m \in f_\star\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, we define $a \cdot m = f^\sharp(a) \cdot m$, where $f^\sharp(a) \in \mathcal{O}_X(f^{-1}(U))$.

Conversely, if $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, we define

$$f^\star\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

where the $f^{-1}\mathcal{O}_Y$-module structure on $\mathcal{O}_X$ is defined via the adjoint to $f^\sharp$.

5.2 Quasi-coherence

Definition. A *quasi-coherent sheaf* $\mathcal{F}$ on a scheme X is a sheaf of $\mathcal{O}_X$-modules such that there exists a cover of X by affines $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is the sheaf associated to a module over the ring $\mathcal{O}_X(U_i)$. If these modules can be taken to be finitely generated, we say $\mathcal{F}$ is *coherent*.

Example. (i) On any scheme X, $\mathcal{O}_X$ is quasi-coherent (and, in fact, coherent).

(ii) $\bigoplus_I \mathcal{O}_X$ is quasi-coherent, but not coherent if I is infinite.

(iii) If $i : X \to Y$ is a closed immersion, then $i_\star \mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Let $U \subseteq Y$ be an affine open set, so $U = \operatorname{Spec} A$. Then $X \cap U \to U$ gives an ideal $I \subseteq A$ which is the kernel of the surjection $\mathcal{O}_Y(U) \to \mathcal{O}_X(X \cap U)$. On U, $i_\star \mathcal{O}_X|_U$ is the sheaf associated to the A-module A/I.

Proposition. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for any affine open $U \subseteq X$ with $U = \operatorname{Spec} A$, $\mathcal{F}|_U$ is the sheaf associated to a module over A.

We first prove the following key technical lemma.

Lemma. Let $X = \operatorname{Spec} A$, $f \in A$, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{F})$. Then
(i) If s restricts to 0 on U_f, then $f^n s = 0$ for some $n \geq 1$.
(ii) If $t \in \mathcal{F}(U_f)$, then $f^n t$ is the restriction of a global section of $\mathcal{F}$ over X for some $n \geq 1$.

Proof. There exists some cover of X by schemes of the form $\operatorname{Spec} B = V$, such that $\mathcal{F}|_V = M^{\mathrm{sh}}$ for M a B-module. We can cover each such V by distinguished affines of the form U_g for some $g \in A$. Then $\mathcal{F}|_{U_g} = (M \otimes_B A_g)^{\mathrm{sh}}$, as $F|_V$ is quasi-coherent. But recall that $\operatorname{Spec} A$ is quasi-compact: every open cover has a finite subcover. So finitely many U_{g_i} will suffice to cover X by open sets such that $\mathcal{F}$ restricts to M_i^{sh} on U_{g_i}. Then the lemma follows from formal properties of localisation. □

We now prove the main proposition.

Proof. Given $U \subseteq X$, observe that $\mathcal{F}|_U$ is also quasi-coherent. We can thus reduce the statement to the case where $X = \operatorname{Spec} A$. Now we take $M = \Gamma(X, \mathcal{F})$, and let M^{sh} be the associated sheaf. We claim that $M^{\mathrm{sh}} \cong \mathcal{F}$. Let $\alpha : M^{\mathrm{sh}} \to \mathcal{F}$ be the map given by restriction (for example via stalks). Then α is an isomorphism at the level of stalks by the above lemma, so is an isomorphism globally. □

In particular, the quasi-coherent sheaves of modules over $\operatorname{Spec} A$ are precisely the modules over A. The coherent sheaves of modules over $\operatorname{Spec} A$ are precisely the finitely-generated modules over A.

Proposition. (i) Images, kernels, and cokernels of maps of (quasi-)coherent sheaves remain (quasi-)coherent.
(ii) If $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a (quasi-)coherent sheaf of modules on S, then $f^\star \mathcal{F}$ is also (quasi-)coherent.

(iii) If $f : X \to S$ is a morphism of schemes and $\mathcal{G}$ is a quasi-coherent sheaf on X, then $f_\star \mathcal{G}$ is also quasi-coherent.

The proofs are omitted and non-examinable. Note that (iii) need not hold for coherent sheaves: let $f : \mathbb{A}^1_k \to \operatorname{Spec} k$ be the obvious map, and consider $f_\star \mathcal{O}_{\mathbb{A}^1_k}$. This is a quasi-coherent sheaf on $\operatorname{Spec} k$, so is a k-vector space, which is $k[t]$. As a module, this is not finitely generated. Observe that if $f : \mathbb{P}^1_k \to \operatorname{Spec} k$, then $f_\star \mathcal{O}_{\mathbb{P}^1_k}$ is the sheaf associated to k. In general, if $\mathcal{G}$ is a coherent sheaf on X and $f : X \to S$ is proper, then $f_\star \mathcal{G}$ is coherent.

Let A be a graded ring, with the usual assumptions on its generators. To build $\operatorname{Proj} A$, we consider the cover by $\operatorname{Spec}\left(A\left[\frac{1}{f}\right]_0\right)$ for $f \in A_1$. We can produce a similar construction for modules.

Let M be a *graded A-module*, that is,

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

where each M_d is an abelian group, M is an A-module, and $A_i M_j \subseteq M_{i+j}$. Consider the sheaf determined by the association

$$\operatorname{Proj} A \supseteq U_f \mapsto \left(M\left[\frac{1}{f}\right]\right)_0$$

To each $U_f = \mathbb{V}(f)^c$, we associate the degree zero elements of the localisation of M at f. This gives a quasi-coherent sheaf on $\operatorname{Proj} A$ by identical arguments as in the Proj construction.

Definition. Let X be a scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is
(i) *free*, if $\mathcal{F} \simeq \mathcal{O}_X^{\oplus I}$ for some set I;
(ii) an *(algebraic) vector bundle* or *locally free* if there exists an open cover $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is free;
(iii) a *line bundle* or an *invertible sheaf* if it is a vector bundle that is locally isomorphic to $\mathcal{O}_X$.

Note that such sheaves are coherent if and only if the index sets I can be taken to be finite.

5.3 Coherent sheaves on projective schemes

Definition. Let A be a graded ring, and let M be a graded A-module. For $d \in \mathbb{Z}$, we define $M(d)$, called M *twisted by* d, to be the module such that

$$(M(d))_k = M_{k+d}$$

Definition. Let $X = \operatorname{Proj} A$ where A is a graded ring and let $d \in \mathbb{Z}$. The sheaf $\mathcal{O}_X(d)$ is defined to be the sheaf associated to the graded module $A(d)$. In particular, $\mathcal{O}_X(1)$ is called the *twisting sheaf.*

Remark. $\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$. Note that the tensor product of graded modules is additive in the grading.

Example. Consider $\operatorname{Proj} k[x_0, \dots, x_n] = \mathbb{P}^n_k$. The global sections of $\mathcal{O}_{\mathbb{P}^n_k}(d)$ are homogeneous degree d polynomials in the x_i. In particular, if $d < 0$, then $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) = 0$.

Definition. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *globally generated* or *generated by global sections* if it is a quotient of $\mathcal{O}_X^{\oplus r}$ for some r; that is, is there is a surjective map of coherent sheaves $\mathcal{O}_X^{\oplus r} \to \mathcal{F}$. Equivalently, there exist elements $s_1, \dots, s_r \in \Gamma(X, \mathcal{F})$ such that $\{s_i\}$ generate the stalks $\mathcal{F}_p$ over $\mathcal{O}_{X,p}$ for all $p \in X$.

Theorem. Let $i : X \rightarrowtail \mathbb{P}^n_R$ be a closed immersion. Let $\mathcal{O}_X(1)$ be the restriction of $\mathcal{O}_{\mathbb{P}^n_R}(1)$, so $\mathcal{O}_X(1) = i^\star \mathcal{O}_{\mathbb{P}^n_R}(1)$. Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists an integer d_0 such that for all $d \geq d_0$, the sheaf

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$$

is globally generated.

Proof. By formal properties, it is equivalent to show the statement for $i_\star\mathcal{F}$; that is, $i_\star\mathcal{F}(d)$ is globally generated on $\mathbb{P}^n_R$. Write $\mathbb{P}^n_R = \operatorname{Proj}[x_0, \dots, x_n]$, and cover $\mathbb{P}^n_R$ by $U_i = \operatorname{Spec} B_i$ where $B_i = R\left[\frac{x_0}{x_i}\right]$. We know that $\mathcal{F}|_{U_i} = M_i^{\mathrm{sh}}$, and M_i is a finitely generated B_i-module. Let $\{s_{ij}\}$ be generators for M_i. We claim that the sections $\{x_i^d s_{ij}\}_j$ of $\mathcal{F}(d)|_{U_i}(U_i)$ are restrictions of global sections t_{ij} of $\mathcal{F}(d)$ for sufficiently large d. Such d can be chosen to be independent of i and j. Indeed, if s_{ij} is an element of $M_i = \mathcal{F}(U_i)$ and $x_i \in \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n_r}(1)$, we can show that $x_i^d s_{ij} \in (F \otimes \mathcal{O}(d))(U_i)$ is a restriction of a global section.

Now, on U_i, the s_{ij} generate M_i^{sh}, but we have a morphism of sheaves $\mathcal{F} \to \mathcal{F}(d)$, mapping s to $x_i^d s \coloneqq s \otimes x_i^d$. This map is globally defined, but on U_i this restricts to an isomorphism $\mathcal{F}|_{U_i} \to \mathcal{F}(d)|_{U_i}$ as x_i is invertible on U_i. Since the $\{s_{ij}\}$ generate $\mathcal{F}|_{U_i}$, the $x_i^d s_j$ generate $\mathcal{F}(d)|_{U_i}$. Thus, the t_{ij} globally generate $\mathcal{F}(d)$. $\square$

Corollary. Let $i : X \rightarrowtail \mathbb{P}^n_R$ be a closed immersion. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is a quotient of $\mathcal{O}(-d)^{\oplus N}$ for some sufficiently large N and some $d \in \mathbb{Z}$.

6 Divisors

6.1 Height and dimension

Recall that for a prime ideal $\mathfrak{p}$ in R, its *height* is the largest n such that there exists a chain of inclusions of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

For example, if R is an integral domain, a prime ideal is of height 1 if and only if no nonzero prime ideal is strictly contained within it.

Example. (i) In any integral domain, (0) has height 0.

(ii) In $\mathbb{C}[x, y]$, the ideal (x) has height 1, and the ideal (x, y) has height 2.

It can be shown that in a unique factorisation domain, every prime ideal of height 1 is principal.

We will globalise the notion of height 1 prime ideals, giving *Weil divisors*, and also the notion of principal ideals, giving *Cartier divisors*. In the case of Weil divisors, we will assume that the ambient scheme X is Noetherian, integral, separated, and *regular in codimension 1*.

If X is integral and $U = \operatorname{Spec} A$ is an open affine, then the ideal $(0) \subseteq A$ is called the *generic point* of X. Each open affine is dense as they are irreducible, so they have a nontrivial intersection, including their generic points. The generic points given by each U therefore coincide in X. This point is often denoted by η or η_X.

Definition. Let X be a scheme.

(i) The *dimension* of X is the length n of the longest chain of nonempty closed irreducible subsets
$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$
(ii) Let $Z \subseteq X$ be closed and irreducible. The *codimension* of X is the length n of the longest chain
$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$
(iii) If X is a *Noetherian topological space*, so every decreasing sequence of closed subsets stabilises, then every closed $Z \subseteq X$ has a decomposition into finitely many irreducible closed subsets.

(iv) Suppose X is Noetherian, integral, and separated. We say that X is *regular in codimension 1* if for every subspace $Y \subseteq X$ that is closed, irreducible, and of codimension 1, if η_Y denotes the generic point of Y, then $\mathcal{O}_{X,\eta_Y}$ is a discrete valuation ring, or equivalently a local principal ideal domain.

6.2 Weil divisors

Definition. Let X be Noetherian, integral, separated, and regular in codimension 1. A *prime divisor* on X is an integral closed subscheme of codimension 1. A *Weil divisor* on X is an element of the free abelian group $\operatorname{Div}(X)$ generated by the prime divisors.

We will write $D \in \operatorname{Div}(X)$ as $\sum_i n_{Y_i}[Y_i]$ where the Y_i are prime divisors.

Definition. A Weil divisor $\sum_i n_{Y_i}[Y_i]$ is *effective* if all n_{Y_i} are nonnegative.

If X is integral, for $\operatorname{Spec} A = U \subseteq X$, the local ring $\mathcal{O}_{X,\eta}$ is a field, as it is in particular the fraction field of A. Indeed, because η is contained in every open affine, $\mathcal{O}_{X,\eta}$ permits arbitrary denominators.

Let $f \in \mathcal{O}_{X,\eta_X} = k(X)$ be nonzero. Since for every prime divisor $Y \subseteq X$, the ring $\mathcal{O}_{X,\eta_Y}$ is a discrete valuation ring, we can calculate the valuation $\nu_Y(f)$ of f in this ring. We thus define the divisor
$$\operatorname{div}(f) = \sum_{Y \subseteq X \text{ prime}} \nu_Y(f)[Y]$$
We claim that this is a Weil divisor; that is, the sum is finite.

Proposition. The sum
$$\sum_{Y \subseteq X \text{ prime}} \nu_Y(f)[Y]$$
is finite.

Proof. Let $f \in k(X)^\times$, and choose A such that $U = \operatorname{Spec} A$ is an affine open, so $FF(A) = k(X)$. We can also require that $f \in A$ by localising at the denominator, so f is *regular* on U. Then $X \setminus U$ is closed and of codimension at least 1, so only finitely many prime Weil divisors Y of X are contained in $X \setminus U$. On U, as f is regular, $\nu_Y(f) \geq 0$ for all Y. But $\nu_Y(f) > 0$ if and only if Y is contained in $\mathbb{V}(f) \subseteq U$, and by the same argument, there are only finitely many such Y. □

Definition. A Weil divisor of the form $\operatorname{div}(f)$ is called *principal*. In $\operatorname{Div}(X)$, the set of principal divisors form a subgroup $\operatorname{Prin}(X)$, and we define the *Weil divisor class group* of X to be
$$\operatorname{Cl}(X) = {}^{\operatorname{Div}(X)}\!/_{\operatorname{Prin}(X)}$$

Remark. (i) Let A be a Noetherian domain. Then A is a unique factorisation domain if and only if A is integrally closed and $\operatorname{Cl}(\operatorname{Spec} A)$ is trivial. This is related to the fact that in unique factorisation domains, all primes of height 1 are principal. In particular, there exist rings with nontrivial class groups of their spectra.

(ii) $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.

(iii) $\operatorname{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$; we will prove this shortly.

(iv) Let $Z \subseteq X$ is closed, and let $U = X \setminus Z$. Then there is a surjective map $\operatorname{Cl}(X) \twoheadrightarrow \operatorname{Cl}(U)$, defined by $[Y] \mapsto [Y \cap U]$, but instead mapping $[Y]$ to zero if $Y \cap U = \varnothing$. This is well-defined, as $k(X)$ and $k(U)$ are naturally isomorphic, so principal divisors are mapped to principal divisors. For surjectivity, note that given a prime Weil divisor $D \subseteq U$, its closure $\overline{D}$ in X is a prime Weil divisor that restricts to D under the map.

(v) If Z has codimension at least 2, then $\operatorname{Cl}(X) \twoheadrightarrow \operatorname{Cl}(U)$ is an isomorphism. This is because Z does not enter the definition of $\operatorname{Cl}(X)$.

(vi) If $Z \subseteq X$ is integral, closed, and of codimension 1, there is an exact sequence
$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0$$
called the *excision* exact sequence. Indeed, the kernel of $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ are exactly the divisors in X contained in Z.

Proposition. Let k be a field. Then, $\operatorname{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$.

Proof. Let $D \subseteq \mathbb{P}^n$ be integral, closed, and of codimension 1. Then $D = \mathbb{V}(f)$ where f is homogeneous of some degree d; we will define $\deg(D) = d$. We extend linearly to obtain a homomorphism $\deg : \operatorname{Div}(\mathbb{P}^n_k) \to \mathbb{Z}$. We claim that this gives an isomorphism $\operatorname{Cl}(\mathbb{P}^n_k) \to \mathbb{Z}$. First, this is well defined on classes, since if $f = \frac{g}{h}$ is a rational function, then g and h are homogeneous polynomials of the same degree, so $\deg(\operatorname{div}(f)) = 0$. This is surjective, by taking $H = \mathbb{V}(x_0)$ for x_0 homogeneous linear. For injectivity, suppose $D = \sum n_{Y_i}[Y_i]$ with $\sum n_{Y_i} \deg(Y_i) = 0$. Write $Y_i = \mathbb{V}(g_i)$, and let $f = \prod g_i^{n_{Y_i}}$. Now f is a homogeneous rational function of degree zero. □

6.3 Cartier divisors

Let X be a scheme. Consider the presheaf on X given by mapping $U = \operatorname{Spec} A$ to $S^{-1}A$ where S is the set of all elements that are not zero divisors. Sheafification yields the sheaf of rings $\mathcal{K}_X$. Define $\mathcal{K}_X^\star \subseteq \mathcal{K}_X$ to be the subsheaf of invertible elements; this is a sheaf of abelian groups under multiplication. If X is integral, then $\mathcal{K}_X$ is the constant sheaf, where the constant field is $\mathcal{O}_{X,\eta_X} = FF(A)$ for any affine open $\operatorname{Spec} A$.

Similarly, let $\mathcal{O}_X^\star \subseteq \mathcal{O}_X$ be the subsheaf of invertible elements. Thus, every section of ${}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}$ can be prescribed by $\{(U_i, f_i)\}$ where U_i is a cover of X, f_i is a section of $\mathcal{K}_X^\star(U_i)$, and that on $U_i \cap U_j$, the ratio ${}^{f_i}\!/\!_{f_j}$ lies in $\mathcal{O}_X^\star(U_i \cap U_j)$.

Definition. A *Cartier divisor* is a global section of the sheaf ${}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}$.

We have a surjective sheaf homomorphism $\mathcal{K}_X^\star \to {}^{\mathcal{K}_\star^X}\!/\!_{\mathcal{O}_X^\star}$, but a global section of ${}^{\mathcal{K}_\star^X}\!/\!_{\mathcal{O}_X^\star}$ is not necessarily the image of a global section of $\mathcal{K}_X^\star$.

Definition. The image of $\Gamma(X, \mathcal{K}_X^\star)$ in $\Gamma\left(X, {}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}\right)$ is the set of *principal* Cartier divisors. The *Cartier class group* is the quotient

$$\Gamma\left(X, {}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}\right)\Big/_{\operatorname{im}\Gamma(X, \mathcal{K}_X^\star)}$$

A section $\mathcal{D} \in \Gamma\left(X, {}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}\right)$ can be specified by $\{(U_i, f_i)\}$ where the $\{U_i\}$ form an open cover and $f_i \in \mathcal{K}_X^\star(U_i)$, such that on $U_i \cap U_j$, the quotient $\frac{f_i}{f_j}$ lies in $\mathcal{O}_X^\star(U_i \cap U_j)$.

Let X be Noetherian, integral, separated, and regular in codimension 1. Given a Cartier divisor $\mathcal{D} \in \Gamma\left(X, {}^{\mathcal{K}_X^\star}\!/\!_{\mathcal{O}_X^\star}\right)$, we obtain a Weil divisor as follows. If $Y \subseteq X$ is a prime Weil divisor and its generic point is η_Y, we represent $\mathcal{D}$ by $\{(U_i, f_i)\}$ and set n_Y to be $\nu_Y(f_i)$ for some U_i containing η_Y. Then we obtain the Weil divisor

$$\sum_{Y \subseteq X} n_Y[Y]$$

This is well-defined: if η_Y is contained in both U_i and U_j, the valuations of f_i and f_j differ by $\nu_Y\left(\frac{f_i}{f_j}\right)$, but $\frac{f_i}{f_j}$ is a unit, so has valuation zero. Similarly, one can show that this is independent of the choice of representative of $\mathcal{D}$.

Proposition. Let X be Noetherian, integral, separated, and regular in codimension 1. Suppose that all local rings $\mathcal{O}_{X,x}$ are unique factorisation domains. Then the association of a

Weil divisor to each Cartier divisor is a bijection, and furthermore, is a bijection of principal divisors.

Proof sketch. If R is a unique factorisation domain, then all height 1 prime ideals are principal. If $x \in X$, then $\mathcal{O}_{X,x}$ is a unique factorisation domain by hypothesis, so given a Weil divisor D, we can restrict it to $\operatorname{Spec} \mathcal{O}_{X,x} \to X$. But on $\operatorname{Spec} \mathcal{O}_{X,x}$, D is given by $\mathbb{V}(f_x)$ as $\mathcal{O}_{X,x}$ is a unique factorisation domain. f_x extends to some neighbourhood U_x containing x, then the f_x can be glued to form a Cartier divisor. This can be checked to be bijective. □

Given a Cartier divisor D on X with representative $\{(U_i, f_i)\}$, we can define $L(\mathcal{D}) \subseteq \mathcal{K}_X$ to be the sub-$\mathcal{O}_X$-module generated on U_i by f_i^{-1}. Note that if $X = \operatorname{Spec} A$ where A is integral, and $\mathcal{D} = \{(X, f)\}$ where $f \in A$, then $A_f \subseteq FF(A)$ is an A-module.

Proposition. The sheaf $L(\mathcal{D})$ is a line bundle.

Proposition. On U_i, we have an isomorphism $\mathcal{O}_{U_i} \to L(\mathcal{D})|_{U_i}$ given by $1 \mapsto f_i^{-1}$.

Consider $X = \mathbb{P}^n_k$, and let D be the Weil divisor $\mathbb{V}(x_0)$. Let $\mathcal{D}$ be the corresponding Cartier divisor. One can show that $\mathcal{O}_{\mathbb{P}^n_k}(1) \cong L(\mathcal{D})$.

Remark. A line bundle L on X has an 'inverse' under the tensor product; that is, defining $L^{-1} = \operatorname{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$, we obtain $L \otimes_{\mathcal{O}_X} L^{-1} = \mathcal{O}_X$. Tensor products of line bundles are also line bundles. If all Weil divisors are Cartier, then $L(\mathcal{D} + \mathcal{E}) = L(\mathcal{D}) \otimes L(\mathcal{E})$.

Definition. The *Picard group* of X is the set of line bundles on X up to isomorphism, which forms an abelian group under the tensor product.

Under mild assumptions, for example assuming that X is integral, the map $\mathcal{D} \mapsto L(\mathcal{D})$ is surjective, and the kernel is exactly the set of principal Cartier divisors.

7 Sheaf cohomology

7.1 Introduction and properties

We have previously seen that if $X = \mathbb{A}^2 \setminus \{(0,0)\}$, then $\mathcal{O}_X(X) \cong \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \cong k[x,y]$. Given a topological space X and a sheaf $\mathcal{F}$ of abelian groups, there is a series of *cohomology* groups $H^i(X, \mathcal{F})$ for $i \in \mathbb{N}$. The definition will be omitted. These groups have the following features.

(i) The group $H^0(X, \mathcal{F})$ is precisely $\Gamma(X, \mathcal{F})$.

(ii) If $f : Y \to X$ is continuous, there is an induced map $f^\star : H^i(X, \mathcal{F}) \to H^i(Y, f^{-1}\mathcal{F})$.

(iii) Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

we obtain a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{F}') \arrow[r] & H^0(X,\mathcal{F}'') \arrow[dll] \\
& H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}'') \arrow[dll] \\
& H^2(X,\mathcal{F}) \arrow[r] & \cdots &
\end{tikzcd}$$

(iv) If X is an affine scheme and $\mathcal{F}$ is a quasi-coherent sheaf, then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.

(v) Cohomology commutes with taking direct sums of sheaves.

(vi) If X is a Noetherian separated scheme, then $H^i(X,\mathcal{F})$ can be computed from the sections of $\mathcal{F}$ on an open affine cover $\{U_i\}$ and from the data of the restrictions to $\mathcal{F}(U_i \cap U_j)$, $\mathcal{F}(U_i \cap U_j \cap U_k)$ and so on. This can be done by considering *Čech cohomology*.

7.2 Čech cohomology

Let X be a topological space, and let $\mathcal{F}$ be a sheaf on X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a fixed open cover of X, indexed by a well-ordered set I. In this course, we will take $I = \{1,\dots,N\}$, and write $U_{i_0\dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. Čech cohomology attaches data to the triple $(X,\mathcal{F},\mathcal{U})$. The group of *Čech p-cochains* is

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\dots<i_p} \mathcal{F}(U_{i_0\dots i_p})$$

There is a *differential*

$$d : C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$$

where the $i_0,\dots,i_{p+1}$ component of $d\alpha$ is given by

$$(d\alpha)_{i_0\dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \, \alpha_{i_0\dots\hat{i}_k\dots i_{p+1}}\Big|_{U_{i_0\dots i_{p+1}}}$$

where $\hat{i}_k$ denotes that the element i_k of the sequence is omitted. One can easily show that $d^2 : C^p \to C^{p+2}$ is the zero map. Thus, $\{C^p(\mathcal{U},\mathcal{F})\}_p$ has the structure of a *cochain complex.*

Definition. The *ith Čech cohomology* of $(X,\mathcal{F},\mathcal{U})$ is the ith cohomology group of the cochain complex:

$$\check{H}^i(X,\mathcal{F}) = \frac{\ker(C^i(\mathcal{U},\mathcal{F}) \xrightarrow{d} C^{i+1}(\mathcal{U},\mathcal{F}))}{\operatorname{im}(C^{i-1}(\mathcal{U},\mathcal{F}) \xrightarrow{d} C^i(\mathcal{U},\mathcal{F}))}$$

Example. Let $X = S^1$ be the usual circle. Let $\mathcal{F}$ be the constant sheaf $\underline{\mathbb{Z}}$; on any connected open set this sheaf has value $\mathbb{Z}$, and for a general open set with n connected components, this sheaf has value $\mathbb{Z}^n$. Let $\mathcal{U} = \{U,V\}$ where U,V are obtained by deleting disjoint closed intervals from the circle, giving an open cover with $U,V \cong \mathbb{R}$. We have

$$C^0(\mathcal{U},\underline{\mathbb{Z}}) = \mathbb{Z}^2$$

as there is one copy of $\mathbb{Z}$ for U and one for V. Also,

$$C^1(\mathcal{U}, \underline{\mathbb{Z}}) = \mathbb{Z}^2$$

given by $\underline{\mathbb{Z}}(U \cap V)$. The differential is $(a, b) \mapsto (b - a, b - a)$, so

$$\check{H}^0(\mathcal{U}, \underline{\mathbb{Z}}) \cong \mathbb{Z} = \ker d$$

and

$$\check{H}^1(\mathcal{U}, \underline{\mathbb{Z}}) \cong \mathbb{Z} = \operatorname{coker} d$$

Remark. (i) These Čech cohomology groups are equal to the corresponding singular cohomology groups of S^1.

(ii) Note that $\check{H}$ is typically only well-behaved when $\mathcal{U}$ is also well-behaved. That is, $\check{H}^i(\mathcal{U}, \mathcal{F})$ depends on $\mathcal{U}$ and not just X. In the example above, we could have chosen $\mathcal{U} = \{S^1\}$, and in this case, $\check{H}^1(\mathcal{U}, \underline{\mathbb{Z}}) = 0$. Also note that $\underline{\mathbb{Z}}$ is not a quasi-coherent sheaf.

(iii) Let $X = \mathbb{P}^1_k$, $U = X \setminus \{0\}$, $V = X \setminus \{\infty\}$, $\mathcal{U} = \{U, V\}$. Then

$$\check{H}^0(\mathcal{U}, \mathcal{O}_X) = k; \quad \check{H}^1(\mathcal{U}, \mathcal{O}_X) = 0$$

(iv) Let X be Noetherian and separated, and let $\{U_i\}_{i \in I}$ be an affine cover of X, so all $U_{i_0 \dots i_p}$ are affine. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong H^p(X, \mathcal{F})$$

and the isomorphism is natural. Thus, in this particular case, the cohomology is easy to calculate by going via Čech cohomology.

Theorem. Let $X = \mathbb{P}^n_k$ and $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n_k}(d)$. Then there are isomorphisms of graded k-vector spaces
(i) $H^0(X, \mathcal{F}) \cong k[x_0, \dots, x_n]$;
(ii) $H^n(X, \mathcal{F}) \cong \frac{1}{x_0 \dots x_n} k[x_0^{-1}, \dots, x_n^{-1}]$;
(iii) $H^p(X, \mathcal{F}) = 0$ for $p \neq 0, n$.
In particular, $H^0(\mathbb{P}^n_k, \mathcal{O}(d))$ has dimension $\binom{n+d}{d}$, and $H^n(\mathbb{P}^n_k, \mathcal{O}(d))$ has dimension $\binom{-d-1}{n}$.

Proof. We prove this result using Čech cohomology. Part (i) follows from earlier discussions, as $H^0(X, \mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^n_k, \mathcal{O}(d))$.

Part (ii). Consider the standard cover $\mathcal{U}$ of $\mathbb{P}^n_k$ by affines $U_i = \mathbb{V}(x_i)^c$. Observe that

$$\mathcal{F}(U_{i_0 \dots i_p}) = k[x_0, \dots, x_n]_{x_{i_0} \dots x_{i_p}}$$

This k-module is spanned by monomials $x_0^{k_0} \dots x_n^{k_n}$ where $k_{i_0}, \dots, k_{i_p} \in \mathbb{Z}$ and the other coefficients are nonnegative. In the associated Čech complex, we have

$$\check{C}^{n-1} = \bigoplus_{i=0}^{n} k[x_0, \dots, x_n]_{x_0 \dots \hat{x}_i \dots x_n}; \quad \check{C}^n = k[x_0, \dots, x_n]_{x_0 \dots x_n}$$

Since $\mathcal{U}$ contains only $n+1$ elements, $\check{C}^{n+1}$ vanishes. Thus,

$$\begin{aligned} H^n(\mathbb{P}^n_k, \mathcal{F}) &= \check{H}^n(\mathcal{U}, \mathcal{F}) \\ &= \frac{\check{C}^n}{\operatorname{im}(\check{C}^{n-1} \to \check{C}^n)} \\ &= \frac{\operatorname{span}_k \left\{ x_0^{k_0} \dots x_n^{k_n} \mid k_i \in \mathbb{Z} \right\}}{\operatorname{span}_k \left\{ x_0^{k_0} \dots x_n^{k_n} \mid \text{at least one } k_i \geq 0 \right\}} \end{aligned}$$

as required.

Part (iii). We will use the long exact sequence associated to a short exact sequence of sheaves and use induction on the dimension *n*. First, observe that $\mathbb{P}^{n-1}_k$ is isomorphic to the closed subscheme $\mathbb{V}(x_0) \subseteq \mathbb{P}^n_k$. Let $i : \mathbb{P}^{n-1}_k \to \mathbb{P}^n_k$ be the inclusion. Recall that $\mathcal{O}_{\mathbb{P}^n_k}(-1) = L(-H)$ where $H = \mathbb{V}(x_0)$. By a result on the example sheets, we obtain the *ideal sheaf sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_\star \mathcal{O}_{\mathbb{P}^{n-1}_k} \longrightarrow 0$$

where the map $\mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}_{\mathbb{P}^n_k}$ is given by multiplication by x_0. This is analogous to the fact that for an ideal I of a ring A, we have a short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

We obtain an associated long exact sequence for the homology. Assuming the result for dimension up to $n-1$, we can break this into three smaller exact sequences.

$$0 \to H^0(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{\cdot x_0} H^0(\mathbb{P}^n_k, \mathcal{F}) \to H^0(\mathbb{P}^{n-1}_k, \mathcal{F}_{\mathbb{P}^{n-1}_k}) \to H^1(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{\cdot x_0} H^1(\mathbb{P}^n_k, \mathcal{F}) \to 0 \tag{a}$$

where $\mathcal{F}_{\mathbb{P}^{n-1}_k} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n-1}_k}(d)$;

$$0 \longrightarrow H^p(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{\cdot x_0} H^p(\mathbb{P}^n_k, \mathcal{F}) \longrightarrow 0 \tag{b}$$

for $1 < p < n-1$; and

$$0 \to H^{n-1}(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{\cdot x_0} H^{n-1}(\mathbb{P}^n_k, \mathcal{F}) \to H^{n-1}(\mathbb{P}^{n-1}_k, \mathcal{F}_{\mathbb{P}^{n-1}_k}) \to H^n(\mathbb{P}^n_k, \mathcal{F}) \xrightarrow{\cdot x_0} H^n(\mathbb{P}^n_k, \mathcal{F}) \to 0 \tag{c}$$

By using (a) and (c), we observe that (b) is also exact for $p = 1$ and $p = n-1$ by explicit computation in the Čech complex. Now, multiplication by x_0 makes $H^p(\mathbb{P}^n_k, \mathcal{F})$ into a $k[x_0]$-module. We will calculate the localisation $H^p(\mathbb{P}^n_k, \mathcal{F})_{x_0}$. As localisation is exact, $H^p(\mathbb{P}^n_k, \mathcal{F})_{x_0} = H^p\big(U_0, \mathcal{F}|_{U_0}\big)$. But the right-hand side vanishes for $p > 0$ as U_0 is affine. Hence, for any $\alpha \in H^p(\mathbb{P}^n_k, \mathcal{F})$, there exists k such that $x_0^k \alpha = 0$. But multiplication by x_0 is an isomorphism on cohomology by (b), so in fact $H^p(\mathbb{P}^n_k, \mathcal{F}) = 0$ for all $1 \leq p \leq n-1$. □

Given the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow i_\star \mathcal{O}_{\mathbb{P}^{n-1}_k} \longrightarrow 0$$

taking the tensor product with $\mathcal{O}_{\mathbb{P}^n_k}(d)$, one can show that we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(d) \longrightarrow i_\star \mathcal{O}_{\mathbb{P}^{n-1}_k}(d) \longrightarrow 0$$

Note that $\mathcal{O}_{\mathbb{P}^n_k}(d)$ is locally free.

Let X be proper over $\operatorname{Spec} k$ and let $\mathcal{F}$ be a coherent sheaf on X.

Remark. (i) We have observed that $H^0(X,\mathcal{F})$ is a finite-dimensional k-vector space. The same holds for all $H^p(X,\mathcal{F})$.

(ii) If X has dimension n, then $H^p(X,\mathcal{F})$ vanishes for $p > n$. Thus, given $(X,\mathcal{F})$, there are finitely many numbers $h^p(X,\mathcal{F}) = \dim_k H^p(X,\mathcal{F})$.

Definition. The *Euler characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \sum_{p=0}^{\infty} (-1)^p h^p(X,\mathcal{F})$$

Suppose that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of such sheaves. Then the associated long exact sequence gives

$$\chi(F') = \chi(F) + \chi(F'')$$

7.3 Choice of cover

Given a Noetherian separated scheme X, a quasi-coherent sheaf $\mathcal{F}$ on X, and an open affine cover $\mathcal{U}$ which we typically take to be finite, we can construct the Čech cohomology $\check{H}^i(\mathcal{U},\mathcal{F})$. In this subsection, we show that the Čech cohomology is independent of the choice of cover in this case.

Theorem. Let X be affine and let $\mathcal{F}$ be quasi-coherent. For any finite cover $\mathcal{U}$ of X by affine opens, the groups $\check{H}^i(\mathcal{U},\mathcal{F})$ vanish for $i > 0$.

Proof. Define the 'sheafified' Čech complex as follows.

$$\mathcal{C}^p(\mathcal{F}) = \prod_{i_0 < \dots < i_p} i_\star \left.\mathcal{F}\right|_{U_{i_0 \dots i_p}}$$

where $i : U_{i_0 \dots i_p} \to X$ is the inclusion. Then the $\mathcal{C}^p(\mathcal{F})$ are quasi-coherent sheaves. By taking global sections,

$$\Gamma(X, \mathcal{C}^p(\mathcal{F})) = C^p(\mathcal{F})$$

where $C^p(\mathcal{F})$ is the usual group of Čech p-cochains. The same formula used to build the Čech complex gives differentials

$$\mathcal{C}^p(\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{F})$$

as a morphism of sheaves. We intend to show that the usual Čech complex

$$C^0(\mathcal{F}) \longrightarrow C^1(\mathcal{F}) \longrightarrow C^2(\mathcal{F}) \longrightarrow \cdots$$

is exact. By a result on the example sheet, on affines, taking local sections preserves exactness. Thus, it suffices to prove that

$$\mathcal{C}^0(\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{F}) \longrightarrow \mathcal{C}^2(\mathcal{F}) \longrightarrow \cdots$$

is an exact sequence of sheaves. However, the exactness of this sequence can be checked locally on stalks. Let $q \in X$, and suppose $q \in U_j$. Now define the map on stalks $\kappa : \mathcal{C}^p_q(\mathcal{F}) \to \mathcal{C}^{p-1}_q(\mathcal{F})$, where for a cochain α, the $(i_0 \dots i_{p-1})$-component of $\kappa(\alpha)$ is equal to the $(ji_0 \dots i_{p-1})$-component of α, where by convention if $ji_0 \dots i_{p-1}$ is not in increasing order, but $\sigma \in S_{p+1}$ brings it into increasing order and σ has sign -1, we instead take the negation of the component. By direct calculation, one can show that $d\kappa + \kappa d = \mathrm{id}$ on C^p for all p.

We can now verify exactness at each stalk. We know that $\mathrm{im}(\mathcal{C}^{p-1} \to \mathcal{C}^p) \subseteq \ker(\mathcal{C}^p \to \mathcal{C}^{p+1})$. Conversely, if $\alpha \in \ker(\mathcal{C}^p \to \mathcal{C}^{p+1})$, then

$$\alpha = (\kappa d + d\kappa)(\alpha) = d(\kappa\alpha) \in \mathrm{im}(\mathcal{C}^{p-1} \to \mathcal{C}^p)$$

□

Lemma. Let X be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{U} = \{U_1, \dots, U_k\}$ and $\widetilde{\mathcal{U}} = \{U_0, \dots, U_k\}$. That $\check{H}^i(\mathcal{U}, \mathcal{F})$ and $\check{H}^i(\widetilde{\mathcal{U}}, \mathcal{F})$ are naturally isomorphic.

Proof sketch. Let $C^p(\mathcal{F})$ and $\widetilde{C}^p(\mathcal{F})$ be the cochain groups for $\mathcal{U}, \widetilde{\mathcal{U}}$ respectively. There are maps $\widetilde{C}^p(\mathcal{F}) \to C^p(\mathcal{F})$ given by dropping the U_0 data. To make this precise, observe that $\widetilde{\alpha} \in \widetilde{C}^p(\mathcal{F})$ can be viewed as a pair (α, α_0) where $\alpha \in C^p(\mathcal{F})$ and α_0 in C^{p-1} for the sheaf $\mathcal{F}|_{U_0}$ with open cover $\mathcal{U}|_{U_0}$. These maps commute with the differentials, so we have an induced map $\check{H}^i(\widetilde{\mathcal{U}}, \mathcal{F}) \to \check{H}^i(\mathcal{U}, \mathcal{F})$. By reducing to a calculation on the affine U_0, we can deduce using the previous result that this induced map is surjective and injective. □

Corollary. $\check{H}^i(\mathcal{U}, \mathcal{F})$ is independent of the choice of $\mathcal{U}$.

Proof. If $\mathcal{U}, \widetilde{\mathcal{U}}$ are two finite open covers by affines, we can interpolate between them by using $\mathcal{U} \cup \widetilde{\mathcal{U}}$ and use the previous result. □

7.4 Further topics in cohomology

(i) Let $X_d \subseteq \mathbb{P}^3_k$ be the vanishing locus of a homogeneous polynomial f_d of degree $d \neq 2$. Then X_d is not isomorphic to a product over $\operatorname{Spec} k$ of schemes of dimension 1. Conversely, X_2 can be isomorphic to $\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k$, using the Segre embedding. This is a consequence of the sheaf Künneth formula, and in particular, the fact that $h^1(X_d, \mathcal{O}_{X_d}) = 0$.

(ii) The different X_d are non-isomorphic as schemes. This follows from calculating $\chi(X_d)$.

(iii) One next direction in cohomology is *duality theory*. Given a closed immersion $i : Z \subseteq X$, the *ideal sheaf* I_Z is the kernel of the map $i^\star : \mathcal{O}_X \to \mathcal{O}_Z$, which is a coherent sheaf on X. The *conormal sheaf* to the closed immersion i, denoted $N^\vee_{Z/X}$, is given by $i^*\left(I_Z/I_Z^2\right)$, where I_Z^2 is the sheafification of the presheaf $U \mapsto I_Z(U)^2$. If $X \to S$ is separated, then the *cotangent sheaf* is

$$\Omega_{X/S} = N^\vee_{\Delta_{X/S}}$$

A scheme X over $\operatorname{Spec} k$ is called *nonsingular* if Ω_X is locally free. The *dualising sheaf* ω_X is the sheafification of $U \mapsto \bigwedge^{\dim X} \Omega_X(U)$.

Theorem (Serre duality)**.** If X is as above and has dimension n, then if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, there is an isomorphism of cohomology groups

$$H^i(\mathcal{X}, \mathcal{F}) \to H^{n-1}(\mathcal{C}, \mathcal{F}^\vee \otimes \omega_X)^\vee$$

where

$$\mathcal{F}^\vee = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$