Category Theory

Cambridge University Mathematical Tripos: Part III

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1 Definitions and examples

1.1 Categories

Definition. A *category* $\mathcal{C}$ consists of

(i) a collection of *objects* $\mathrm{ob}\,\mathcal{C}$, denoted $A, B, C, \dots$;

(ii) a collection of *morphisms* $\mathrm{mor}\,\mathcal{C}$, denoted $f, g, h, \dots$;

(iii) two operations $\mathrm{dom}, \mathrm{cod} : \mathrm{mor}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{C}$, and we write $f : A \to B$ or $A \xrightarrow{f} B$ to state that f is a morphism with domain A and codomain B;

(iv) an operation $A \mapsto 1_A : A \to A$;

(v) a *composition* operation $(f, g) \mapsto fg : \mathrm{dom}\, g \to \mathrm{cod}\, f$, defined exactly when $\mathrm{cod}\, g = \mathrm{dom}\, f$; satisfying

(vi) $f 1_A = f$ and $1_A g = g$ whenever the composites are defined; and

(vii) $(fg)h = f(gh)$ whenever the composites are defined.

Remark. (i) The collections of objects and morphisms may be sets or classes in some set theory, but our definitions are built to be interpretable in any system supporting first-order logic. If $\mathrm{ob}\,\mathcal{C}$ and $\mathrm{mor}\,\mathcal{C}$ are sets, we call $\mathcal{C}$ a *small* category; otherwise we call it *large*.

(ii) We could formulate a definition of category with no mention of objects, since objects biject with the identity morphisms. We will not take this approach here.

(iii) Note that we choose fg to mean 'first g and then f'; this choice is a convention and the other one may be adopted.

Example. (i) **Set** is the category where the objects are all of the sets, and the morphisms are all of the functions between them, each of which is suitably tagged with an appropriate codomain. This must be done because set-theoretic functions do not 'remember' their codomain: $f(x) = x$ as a function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{C}$ are equal sets.

(ii) **Gp** is the category where the objects are all of the groups, and the morphisms are all of the group homomorphisms.

(iii) **Rng** is the category where the objects are all of the rings, and the morphisms are all of the ring homomorphisms.

(iv) For a field k, $\mathbf{Vect}_k$ is the category where the objects are all of the k-vector spaces, and the morphisms are all of the k-linear maps.

(v) **Top** is the category where the objects are all of the topological spaces, and the morphisms are all of the continuous functions.

(vi) **Met** is the category where the objects are all of the metric spaces, and the morphisms are all of the nonexpansive mappings, i.e. functions that do not increase the distance between points. One could choose a different convention, for example by letting morphisms be arbitrary continuous functions.

(vii) **Mfd** is the category where the objects are all of the smooth manifolds, and the morphisms are C^∞ maps.

(viii) **TopGp** is the category where the objects are all of the topological groups, and the morphisms are the continuous homomorphisms.

(ix) **Htpy** is the category where the objects are all of the topological spaces, and the morphisms are equivalence classes of continuous functions under homotopy.

(x) More generally, if $\simeq$ is an equivalence relation on the morphisms of $\mathcal{C}$ such that $f \simeq g$ implies $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{cod} f = \operatorname{cod} g$, and the relation is stable under composition so $f \simeq g$ implies $fh \simeq gh$ and $kf \simeq kg$, we call $\simeq$ a *congruence*. In this case, we can form the *quotient category* ${}^{\mathcal{C}}\!/_{\simeq}$, which has the same objects as $\mathcal{C}$, but its objects are equivalence classes of morphisms in $\mathcal{C}$ under $\simeq$.

(xi) **Rel** is the category where the objects are all of the sets, and the morphisms $A \to B$ are the relations $R \subseteq A \times B$, where composition is given by

$$S \circ R = \{(a, c) \mid \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}$$

Note that if R and S happen to be functions, $\circ$ is the standard composition operator. Therefore, **Set** is a subcategory of **Rel**.

(xii) **Part** is the category where the objects are all of the sets, and the morphisms $A \to B$ are the partial functions $A \rightharpoonup B$. This is a subcategory of **Rel**, and **Set** is a subcategory of **Part**.

(xiii) Given a category $\mathcal{C}$, we can construct its *opposite category* $\mathcal{C}^{\mathrm{op}}$, where the objects and morphisms are the same as in $\mathcal{C}$, but dom and cod are swapped. We also reverse composition in the opposite category. This gives a duality principle: whenever a statement about categories is proven, a dual statement follows from applying the statement to an opposite category.

(xiv) A small category with one object $\star$ is a *monoid*, a group without inverses. In particular, every group can be seen as a small category on a single object in which every morphism is an isomorphism, i.e. invertible.

(xv) A *groupoid* is a category in which every morphism is an isomorphism. For example, we can construct the *fundamental groupoid* of a topological space X. Here, the objects correspond to points x in X, and represent $\pi_1(X, x)$. Morphisms $x \to y$ are homotopy classes of paths starting at x and ending at y. Composition is path concatenation.

(xvi) A category with at most one morphism between any pair of objects is a *preorder*. The existence of a morphism $A \to B$ corresponds to stating $A \preceq B$ in the preorder. In particular, a partially ordered set (poset) is a small preorder in which the only isomorphisms are identity morphisms.

(xvii) For a field k, $\mathbf{Mat}_k$ is the category where the objects are the natural numbers, and the morphisms $n \to p$ are the $p \times n$ matrices over k. Composition is multiplication of matrices. The identity morphisms are the identity matrices.

1.2 Functors

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a map $\operatorname{ob} \mathcal{C} \xrightarrow{F} \operatorname{ob} \mathcal{D}$ and a map $\operatorname{mor} \mathcal{C} \xrightarrow{F} \operatorname{mor} \mathcal{D}$, such that
(i) $F(\operatorname{dom} f) = \operatorname{dom} Ff$;
(ii) $F(\operatorname{cod} f) = \operatorname{cod} Ff$;
(iii) $F(1_A) = 1_{FA}$; and
(iv) $F(fg) = (Ff)(Fg)$ whenever fg is defined.

Example. (i) The *forgetful functors* $\mathbf{Gp} \to \mathbf{Set}, \mathbf{Rng} \to \mathbf{Set}, \mathbf{Top} \to \mathbf{Set}$ and so on forget that the objects are structures and forget the conditions on morphisms. Similarly, there are forgetful functors $\mathbf{Rng} \to \mathbf{AbGp}, \mathbf{Met} \to \mathbf{Top}, \mathbf{TopGp} \to \mathbf{Top}, \mathbf{TopGp} \to \mathbf{Gp}$.

(ii) Any mapping $f : A \to UG$ from a set A to the underlying set of a group G extends uniquely to a homomorphism $FA \to G$, where FA is the free group on the set A. This can be made into a functor $F : \mathbf{Set} \to \mathbf{Gp}$: given $f : A \to B$, the homomorphism Ff is the unique homomorphism extending $A \xrightarrow{f} B \to FB$. Given $g : B \to C$, then $F(gf)$ and $(Fg)(Ff)$ both extend the same mapping $A \to FC$, so by the uniqueness property they are equal.

(iii) The power-set construction $P : \mathbf{Set} \to \mathbf{Set}$ is a functor. PA is the set of all subsets of A, and given $f : A \to B$, Pf is the map sending S to the image of S under f.

(iv) There is another power-set functor $P^\star : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ (or $\mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$). This has the same object map, but given $f : A \to B$, $P^\star f$ maps $S \subseteq B$ to its inverse image under f. A functor like this that reverses the direction of arrows is sometimes called *contravariant*; functors which do not are called *covariant*.

(v) The construction of dual spaces in linear algebra gives rise to a functor $(-)^\star : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$. $V^\star$ is the space of linear maps $V \to k$, and a linear map $f : V \to W$ gives rise to $f^\star : W^\star \to V^\star$ given by composition.

(vi) $\mathbf{Cat}$ is the category where the objects are the small categories and the morphisms are functors. This is well-defined as functors have identities and compositions.

(vii) The assignment $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ defines a (covariant) functor $\mathbf{Cat} \to \mathbf{Cat}$.

(viii) A functor between monoids is a monoid homomorphism.

(ix) A functor between groups is a group homomorphism.

(x) A functor between posets is an order-preserving map.

(xi) If G is a group, a functor $F : G \to \mathbf{Set}$ defines a set $A = F\star$, together with a collection of endomorphisms of A denoted $a \mapsto g \cdot a$ for each $g \in G$. This collection of endomorphisms is compatible with the identity and composition, so is precisely the definition of a group action or permutation representation of G.

(xii) If G is a group, a functor $F : G \to \mathbf{Vect}_k$ is a k-linear representation of G.

(xiii) The fundamental group of a topological space defines a functor $\pi_1 : \mathbf{Top}_\star \to \mathbf{Gp}$, where $\mathbf{Top}_\star$ is the category of pointed topological spaces.

1.3 Natural transformations

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories, and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. A *natural transformation* $\alpha : F \to G$ is a mapping $\operatorname{ob} \mathcal{C} \to \operatorname{mor} \mathcal{D}$ denoted $A \mapsto \alpha_A$, such that

(i) $\alpha_A : FA \to GA$ for all A; and

(ii) for any morphism $f : A \to B$ in $\mathcal{C}$, the square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

commutes. Such squares are called *naturality* squares.

If we have a natural transformation $\beta : G \to H$, we can define $\beta\alpha$ by $(\beta\alpha)_A = \beta_A\alpha_A$. We therefore have a category $[\mathcal{C}, \mathcal{D}]$ whose objects are the functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are the natural transformations between them.

Example. (i) Given a vector space V, we have a linear map $\alpha_V : V \to V^{\star\star}$ sending $v \in V$ to the map $f \mapsto f(v)$. This is a natural transformation $\alpha : 1_{\mathbf{Vect}_k} \to (-)^{\star\star}$. The naturality squares are of the form

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha_V"'] & W \arrow[d, "\alpha_W"] \\ V^{\star\star} \arrow[r, "f^{\star\star}"'] & W^{\star\star} \end{tikzcd}$$

where

$$\alpha_V(v) = f \mapsto f(v); \quad f^{\star\star}(g)(h) = g(f^\star h) = g(h \circ f)$$

We show the naturality square commutes.

$$\begin{aligned} ((g \mapsto h \mapsto g(h \circ f)) \circ \alpha_V)(v) &= (g \mapsto h \mapsto g(h \circ f))(\alpha_V v) \\ &= (g \mapsto h \mapsto g(h \circ f))(k \mapsto kv) \\ &= h \mapsto (k \mapsto kv)(h \circ f) \\ &= h \mapsto (h \circ f)v \\ &= h \mapsto (h(fv)) \\ &= \alpha_W(fv) \\ &= (\alpha_W \circ f)v \end{aligned}$$

(ii) There is an inclusion from any set A to its free group FA. The map sending a set A to the inclusion $A \to FA$ is a natural transformation $1_{\mathbf{Set}} \to UF$. Naturality is built into the definition of F on morphisms.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ UFA \arrow[r, "UF(f)"'] & UFB \end{tikzcd}$$

(iii) There is a mapping $\alpha_A : A \to PA$ by mapping $a \in A$ to $\{a\} \in PA$. This is a natural transformation $1_{\mathbf{Set}} \to P$, since $Pf\{a\} = \{fa\}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ PA \arrow[r, "Pf"'] & PB \end{tikzcd}$$

(iv) Let $f, g : P \rightrightarrows Q$ be order-preserving maps between posets. Then for $x \leq y$ in P, the naturality square is

$$\begin{tikzcd} fx \arrow[r] \arrow[d, "\alpha_x"'] & fy \arrow[d, "\alpha_y"] \\ gx \arrow[r] & gy \end{tikzcd}$$

In particular, the existence of α_x proves that $fx \leq gx$. Thus a natural transformation $f \to g$ exists if and only if $fx \leq gx$ pointwise for all $x \in P$. Note that every square of morphisms in a poset commutes.

(v) Let $u, v : G \rightrightarrows H$ be group homomorphisms. For $g \in G$, the naturality square is

$$\begin{tikzcd} \star \arrow[r, "ug"] \arrow[d, "\alpha_\star"'] & \star \arrow[d, "\alpha_\star"] \\ \star \arrow[r, "vg"'] & \star \end{tikzcd}$$

A natural transformation $\alpha : u \to v$ is an element $\alpha_\star = h \in H$ such that $hu(g) = v(g)h$ for all g, or equivalently, $v(g) = hu(g)h^{-1}$. Thus a natural transformation exhibits a conjugacy between two homomorphisms. In particular, the natural transformations $u \to u$ are the elements of the centraliser of $u(G)$.

(vi) Let A, B be permutation representations of G, that is, functors $G \to \mathbf{Set}$.

$$\begin{tikzcd} A\star \arrow[r, "Ag"] \arrow[d, "f"'] & A\star \arrow[d, "f"] \\ B\star \arrow[r, "Bg"'] & B\star \end{tikzcd}$$

A natural transformation $f : A \to B$ is a mapping of the underlying sets $A\star \to B\star$ satisfying $g \cdot f(a) = f(g \cdot a)$ for all $a \in A$ and $g \in G$. This is the definition of a G-equivariant map.

(vii) For any (nice) pointed topological space X with base point x, the *Hurewicz homomorphism* is a map $h_{n,x} : \pi_n(X, x) \to H_n(X)$. This is a natural transformation $\pi_n \to H_n U$ where U is the forgetful functor $\mathbf{Top}_\star \to \mathbf{Top}$.

1.4 Equivalence of categories

There is a notion of isomorphism of categories, namely, isomorphism in the category **Cat**. For example, $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$ via the functor

$$A \mapsto A; \quad R \mapsto R^\circ = \{(b, a) \mid (a, b) \in R\}$$

However, there is a weaker notion that is often more useful in practice, called equivalence. To define this, we need a notion of 'natural isomorphism'. There are two obvious definitions, which we show are equivalent.

Lemma. Let $\alpha : F \to G$ be a natural transformation between functors $\mathcal{C} \rightrightarrows \mathcal{D}$. Then α is an isomorphism in the functor category $[\mathcal{C}, \mathcal{D}]$ if and only if each component α_A is an isomorphism in $\mathcal{D}$.

Proof. The forward direction is clear as composition in $[\mathcal{C}, \mathcal{D}]$ is pointwise; if β is an inverse for α, then β_A is an inverse for α_A. Suppose β_A is an inverse for α_A for each A. We show the β collectively form a natural transformation by verifying the naturality squares. Given $f : A \to B$ in $\mathcal{C}$, consider

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, shift left, "\alpha_A"] & FB \arrow[d, shift right, "\alpha_B"'] \\ GA \arrow[r, "Gf"'] \arrow[u, shift left, "\beta_A"] & GB \arrow[u, shift right, "\beta_B"'] \end{tikzcd}$$

Then

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(Gf)\alpha_A\beta_A = \beta_B(Gf)$$

using naturality of α. Thus β is natural, and an inverse for α. □

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories. An *equivalence* between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors

$$F : \mathcal{C} \to \mathcal{D}; \quad G : \mathcal{D} \to \mathcal{C}$$

and a pair of natural isomorphisms

$$\alpha : 1_{\mathcal{C}} \to GF; \quad \beta : FG \to 1_{\mathcal{D}}$$

If $\mathcal{C}$ and $\mathcal{D}$ are equivalent, we write $\mathcal{C} \simeq \mathcal{D}$.

The reason the natural isomorphisms point in opposite directions will be clarified later. A property P of categories that is called *categorical* if whenever $\mathcal{C}$ satisfies P and $\mathcal{C} \simeq \mathcal{D}$, then $\mathcal{D}$ satisfies P. For example, the properties of being a preorder or being a groupoid are categorical. Being a partial order or being a group are not categorical. Generally, properties that rely on equality of objects, not isomorphism, will not be categorical.

Example. (i) Let $\mathbf{Set}_\star$ be the category of pointed sets and functions preserving the base point. Then $\mathbf{Set}_\star \simeq \mathbf{Part}$ by

$$F : \mathbf{Set}_\star \to \mathbf{Part}; \quad F(A, a) = A \setminus \{a\}; \quad F((A, a) \xrightarrow{f} (B, b))(x) = f(x)$$

and

$$G : \mathbf{Part} \to \mathbf{Set}_\star; \quad G(A) = A \cup \{A\}; \quad G(A \xrightarrow{f} B \text{ partial})(x) = \begin{cases} f(x) & \text{if } f \text{ is defined at } x \\ B & \text{otherwise} \end{cases}$$

Note that $FG = 1_{\mathbf{Part}}$, but GF is not equal to $1_{\mathbf{Set}_\star}$. It is not possible for these two categories to be isomorphic, because there is an isomorphism class of **Part** that has only one member, namely $\{\varnothing\}$, but this cannot occur in $\mathbf{Set}_\star$.

(ii) Let $\mathbf{fdVect}_k$ be the category of finite-dimensional vector spaces over k. This category is equivalent to its opposite category $\mathbf{fdVect}_k^{\mathrm{op}}$ via the dual space functors in both directions. The natural isomorphisms α and β are both as in the double dual example given above.

(iii) We show $\mathbf{fdVect}_k \simeq \mathbf{Mat}_k$. Define

$$F : \mathbf{Mat}_k \to \mathbf{fdVect}_k; \quad F(n) = k^n$$

and sending a matrix A to the linear map it represents in the standard basis. For each finite-dimensional vector space V, choose a particular basis. Define

$$G : \mathbf{fdVect}_k \to \mathbf{Mat}_k; \quad G(V) = \dim V$$

and let $G(\theta)$ be the matrix representing θ with respect to the particular bases chosen above. Then $GF = 1_{\mathbf{Mat}_k}$, as long as we chose the bases above in such a way that the k^n have the standard basis. Further, FG is naturally isomorphic to $1_{\mathbf{fdVect}_k}$, since the chosen bases define isomorphisms $k^{\dim V} \to V$, which are natural in V.

In line with the idea that we do not want to consider equality of objects but only equality of morphisms, we make the following definitions.

Definition. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is
(i) *faithful*, if for each $f, g \in \operatorname{mor} \mathcal{C}$ with equal domain and codomain, $Ff = Fg$ implies $f = g$;
(ii) *full*, if for each $FA \xrightarrow{g} FB$, there exists a morphism $A \xrightarrow{f} B$ such that $Ff = g$;
(iii) *essentially surjective*, if every $B \in \operatorname{ob} \mathcal{D}$ is isomorphic to some FA for $A \in \operatorname{ob} \mathcal{C}$.

Note that if F is full and faithful, it is *essentially injective*: if $FA \xrightarrow{g} FB$ is an isomorphism, the unique $A \xrightarrow{f} B$ with $Ff = g$ is an isomorphism, because its inverse is the unique $B \to A$ mapped to g^{-1}.

Lemma. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is part of an equivalence $\mathcal{C} \simeq \mathcal{D}$ if and only if F is full, faithful, and essentially surjective.

Proof. Suppose G, α, β make F into an equivalence. The existence of β ensures that $B \simeq FGB$ for any $B \in \operatorname{ob} \mathcal{D}$, giving essential surjectivity. For faithfulness, for any $A \xrightarrow{f} B$ in $\mathcal{C}$, we have $f = \alpha_B^{-1}(GFf)\alpha_A$, allowing us to reproduce f from its domain, codomain, and image under F. For fullness, consider $FA \xrightarrow{g} FB$, and define $f = \alpha_B^{-1}(Gg)\alpha_A : A \to B$. Then, $GFf = Gg$. As G is faithful by symmetry, $Ff = g$.

For the converse, for each object $B \in \mathcal{D}$, we choose an isomorphism $\beta_B : FA \to B$ where $A \in \mathcal{C}$, and define the action of G at B to be this A. Then we define G on morphisms by letting $G(B \xrightarrow{g} C)$ be the unique $GB \to GC$ whose image under F is $\beta_C^{-1} \circ g \circ \beta_B$, thus making the following diagram commute.

$$\begin{tikzcd} FGB \arrow[r, "FGg"] \arrow[d, "\beta_B"'] & FGC \\ B \arrow[r, "g"'] & C \arrow[u, "\beta_C^{-1}"'] \end{tikzcd}$$

This is functorial: given $h : C \to D$, we can form $G(hg)$ and $(Gh)(Gg)$ which have the same image under F, so must be equal.

$$\begin{tikzcd}
& FGB \arrow[rr, "FG(hg)"] \arrow[dl, "\beta_B"'] \arrow[dr, "FGg"] & & FGD & \\
B \arrow[dr, "g"'] & & FGC \arrow[ur, "FGh"] \arrow[dr, "\beta_C"] & & D \arrow[ul, "\beta_D^{-1}"'] \\
& C \arrow[ur, "\beta_C^{-1}"] \arrow[rr, "1_C"'] & & C \arrow[ur, "h"'] &
\end{tikzcd}$$

By construction, β is a natural isomorphism $FG \to 1_{\mathcal{D}}$. It suffices to construct the natural isomorphism $\alpha : 1_{\mathcal{C}} \to GF$. Its component at A is the unique isomorphism whose image under F is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA$$

Consider a naturality square for α.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\
GFA \arrow[r, "GFf"'] & GFB
\end{tikzcd}$$

As F is faithful, to show this diagram commutes, it suffices to show that its image under F commutes.

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "F\alpha_A=\beta_{FA}^{-1}"'] & FB \arrow[d, "F\alpha_B=\beta_{FB}^{-1}"] \\
FGFA \arrow[r, "FGFf"'] & FGFB
\end{tikzcd}$$

This commutes by naturality of β^{-1}. □

We call a subcategory full if its inclusion functor is full.

Definition. A category is called *skeletal* if every isomorphism class has a single member. A *skeleton* of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ containing exactly one object for each isomorphism class.

Note that an equivalence of skeletal categories is bijective on objects, and hence is an isomorphism of categories.

1.5 Monomorphisms and epimorphisms

Definition. A morphism $f : A \to B$ is a *monomorphism*, and is called *monic*, if $fg = fh$ implies $g = h$ whenever the compositions are defined. Dually, f is an *epimorphism*, and is called *epic*, if $gf = hf$ implies $g = h$ whenever the compositions are defined.

Monomorphisms are left-cancellable; epimorphisms are right-cancellable. We will often denote a monomorphism with an arrow with a tail $A \rightarrowtail B$, and denote epimorphisms with double-headed arrows $A \twoheadrightarrow B$. Isomorphisms are clearly monic and epic; if all monic and epic morphisms in a category are isomorphisms, we call the category *balanced*.

Example. (i) In **Set**, the monomorphisms are precisely the injective functions, and the epimorphisms are precisely the surjective functions. Thus **Set** is balanced.

(ii) In **Gp**, the monomorphisms are the injective functions, and the epimorphisms are the surjective functions.

(iii) In **Rng**, the monomorphisms are again the injective functions, but there are epimorphisms that are not surjective, for example the inclusion $\mathbb{Z} \to \mathbb{Q}$.

(iv) In **Top**, the monomorphisms are the injective functions, and the epimorphisms are the surjective functions. However, **Top** is not balanced, because continuous bijections need not have continuous inverses.

(v) In a preorder, any morphism is monic and epic. The category is balanced if and only if it is an equivalence relation (or equivalently, symmetric).

2 The Yoneda lemma

2.1 Statement and proof

Definition. A category $\mathcal{C}$ is called *locally small* if the collection of morphisms $A \to B$ are parametrised by a set. In this case, we write $\mathcal{C}(A, B)$ for the set of such morphisms.

Given an object A of a locally small category, we can define a functor

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$$

given by

$$B \mapsto \mathcal{C}(A, B); \quad (B \xrightarrow{f} C) \mapsto ((A \xrightarrow{g} B) \mapsto fg)$$

This is functorial by associativity of function composition. We can also define

$$\mathcal{C}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

by

$$B \mapsto \mathcal{C}(B, A); \quad (B \xrightarrow{f} C) \mapsto ((B \xrightarrow{g} A) \mapsto gf)$$

Lemma (Yoneda lemma)**.** Let $\mathcal{C}$ be a locally small category. Let $A \in \mathrm{ob}\,\mathcal{C}$, and let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. Then,

(i) there is a bijection

$$\{\text{natural transformations } \mathcal{C}(A, -) \to F\} \leftrightarrow \{\text{elements of } FA\}$$

(ii) and further, this bijection is natural in both A and F.

This shows that we can consider a natural transformation $\mathcal{C}(A, -) \to F$ as a way to evaluate morphisms at a point $x \in FA$.

Example. Consider the category $\mathcal{C}$ of the form

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C \end{tikzcd}$$

and the functor $F : \mathcal{C} \to \mathbf{Set}$ given by

$$F(A) = \{1, 2\}; \quad F(B) = \{3\}; \quad F(C) = \{4, 5, 6\}$$

and

$$F(f)(1) = F(f)(2) = 3; \quad F(g)(1) = 4; \quad F(g)(2) = 5$$

A natural transformation $\alpha : \mathcal{C}(A, -) \to F$ is given by its components

$$\alpha_A : \{1_A\} \to \{1, 2\}; \quad \alpha_B : \{f\} \to \{3\}; \quad \alpha_C : \{g\} \to \{4, 5, 6\}$$

subject to the naturality square

$$\begin{tikzcd} \{1_A\} \arrow[r, "\mathcal{C}(A,g)"] \arrow[d, "\alpha_A"'] & \{g\} \arrow[d, "\alpha_C"] \\ \{1, 2\} \arrow[r, "Fg"'] & \{4, 5, 6\} \end{tikzcd}$$

which enforces that

$$(Fg)(\alpha_A) = \alpha_C(g)$$

This means that such a natural transformation α is defined uniquely by a choice of $(Fg)(\alpha_A)$; that is, a choice of an element of FA.

Example. Let G be a group in the set-theoretic sense. Let us represent G as the category $\mathcal{C}$; that is, let

$$\operatorname{ob} \mathcal{C} = \{\star\}; \quad \operatorname{mor} \mathcal{C} = G$$

Consider the functor $F : \mathcal{C} \to \mathbf{Set}$ given by

$$F(\star) = G; \quad F(g)(h) = gh$$

If $\alpha : \mathcal{C}(\star, -) \to F$ is a natural transformation, for each $g \in G$, $\alpha_\star(g)$ is a map $G \to G$. The naturality condition ensures that α respects the group structure. Applying the Yoneda lemma, we find that every map $G \to G$ that respects the group structure in this way is just the action of multiplication by some element of the group.

We prove part (i) now, and postpone (ii) until some corollaries have been established.

Proof. We want to show that a natural transformation $\alpha : \mathcal{C}(A, -) \to F$ is a way to evaluate morphisms at a point $x \in FA$. To find a sensible value for x, we evaluate the identity morphism $1_A : A \to A$.

$$\Phi : (\mathcal{C}(A, -) \to F) \to FA; \quad \Phi(\alpha) = \alpha_A(1_A) \in FA$$

Now, given a point $x \in FA$, we want to create a natural transformation that evaluates functions $A \to B$ and yields a point in FB. We define

$$\Psi : FA \to (\mathcal{C}(A, -) \to F); \quad \Psi(x)_B(A \xrightarrow{f} B) = (Ff)x$$

For $h : B \to C$, the naturality square is as follows.

$$\begin{tikzcd} \mathcal{C}(A,B) \arrow[r, "\mathcal{C}(A,h)"] \arrow[d, "\Psi(x)_B"'] & \mathcal{C}(A,C) \arrow[d, "\Psi(x)_C"] \\ FB \arrow[r, "Fh"'] & FC \end{tikzcd}$$

Here, $\mathcal{C}(A,h)$ denotes the operation $g \mapsto hg$. For $f : A \to B$,

$$\Psi(x)_C(\mathcal{C}(A,h)(f)) = \Psi(x)_C(hf) = (F(hf))x$$

and

$$(Fh)(\Psi(x)_B(f)) = (Fh)((Ff)x) = (F(hf))x$$

as required. Hence the 'evaluate at x' map $\Psi(x)$ is a natural transformation. We show that these two constructions are inverses.

$$\Phi\Psi(x) = \Psi(x)_A(1_A) = (F1_A)x = 1_{FA}x = x$$

Let $\alpha : \mathcal{C}(A,-) \to F$ be a natural transformation, let $B \in \operatorname{ob}\mathcal{C}$, and let $f : A \to B$. Then $\alpha_B(f)$ and $(\Psi\Phi(\alpha))_B(f)$ are elements of FB; we show they coincide.

$$(\Psi\Phi(\alpha))_B(f) = (Ff)(\Phi(\alpha)) = (Ff)(\alpha_A(1_A))$$

Naturality of α shows that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C}(A,A) \arrow[r, "\mathcal{C}(A,f)"] \arrow[d, "\alpha_A"'] & \mathcal{C}(A,B) \arrow[d, "\alpha_B"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

Thus,

$$(\Psi\Phi(\alpha))_B(f) = \alpha_B(f1_A) = \alpha_B(f)$$

Hence, Φ and Ψ are inverse bijections. □

Corollary. For any locally small category $\mathcal{C}$, the map

$$A \mapsto \mathcal{C}(A,-)$$

is a full and faithful functor

$$Y : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$$

This is called the *Yoneda embedding*.

Proof. Let $F = \mathcal{C}(B,-)$ in the Yoneda lemma. Then there is a bijection

$$\mathcal{C}(B,A) \leftrightarrow \{\text{natural transformations } \mathcal{C}(A,-) \to \mathcal{C}(B,-)\}$$

This bijection maps $f : B \to A$ to the natural transformation given by composition with f. This is functorial as composition in $\mathcal{C}$ is associative. □

This says that any locally small category $\mathcal{C}$ is equivalent to a full subcategory of a functor category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. The category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is sometimes called the category of *presheaves* on $\mathcal{C}$, so any category embeds into its category of presheaves.

We now explain and prove part (ii) of the Yoneda lemma. Suppose that $\mathcal{C}$ were small, so $[\mathcal{C}, \mathbf{Set}]$ were locally small. Then we have two functors

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$$

The first is the evaluation functor

$$(A, F) = FA$$

The second is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathcal{C}, \mathbf{Set}]^{\mathrm{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

The naturality condition is that Φ and Ψ are natural transformations between these two functors, and thus are natural isomorphisms.

Proof. Let $f : A \to A'$, $\alpha : F \to F'$, and $x \in FA$. If x' is the image of x under the diagonal of the naturality square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FA' \arrow[d, "\alpha_{A'}"] \\ F'A \arrow[r, "F'f"'] & F'A' \end{tikzcd}$$

we want to show that $\Psi(x')$ is the composite

$$\mathcal{C}(A', -) \xrightarrow{\mathcal{C}(f,-)} \mathcal{C}(A, -) \xrightarrow{\Psi(x)} F \xrightarrow{\alpha} F'$$

But this can be easily verified, as the composite maps

$$1_{A'} \mapsto f \mapsto (Ff)(x) \mapsto \alpha_{A'}(Ff)(x) = x'$$

as required. □

2.2 Representable functors

Definition. Let $\mathcal{C}$ be a locally small category. A functor $F : \mathcal{C} \to \mathbf{Set}$ is called *representable* if it is isomorphic to $\mathcal{C}(A, -)$ for some A. A *representation* of F is a pair (A, x) where $A \in \operatorname{ob} \mathcal{C}$, and $x \in FA$ is such that

$$\Psi(x) : \mathcal{C}(A, -) \to F$$

is a natural isomorphism. In this case, we say that x is a *universal element* of F.

Corollary. Suppose (A, x) and (B, y) are representations of $F : \mathcal{C} \to \mathbf{Set}$. Then there is a unique isomorphism $f : A \to B$ such that $Ff(x) = y$.

Proof. The Yoneda lemma shows that the elements of FA correspond to natural transformations $\mathcal{C}(A,-) \to F$, and similarly for the elements of FB. Thus, $Ff(x) = y$ equivalently says that

$$\begin{tikzcd} \mathcal{C}(B,-) \arrow[rr, "\mathcal{C}(f,-)"] \arrow[dr, "\Psi(y)"'] & & \mathcal{C}(A,-) \arrow[dl, "\Psi(x)"] \\ & F & \end{tikzcd}$$

commutes. But $\Psi(x)$ and $\Psi(y)$ are isomorphisms, so this holds if and only if f is the unique isomorphism sent by the Yoneda embedding to $\Psi(x)^{-1}\Psi(y)$. □

(i) Consider the forgetful functor **Gp** $\to$ **Set**. This is representable by the free group on one generator, $\mathbb{Z}$. Similarly, the forgetful functor **Rng** $\to$ **Set** is represented by the free ring on one generator, $\mathbb{Z}[x]$.

(ii) The forgetful functor **Top** $\to$ **Set** is representable by the one-point space.

(iii) The contravariant power set functor $P^\star : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is representable by the two-element set $2 = \{0,1\}$ via the bijection mapping $f : A \to 2$ to $f^{-1}(1)$.

(iv) The covariant power set functor $P : \mathbf{Set} \to \mathbf{Set}$ is not representable. $\mathbf{Set}(A,1) \cong 1$ for any A, but $P1 \cong 2 \ncong 1$.

(v) Define $\Omega : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ to be the functor mapping a space X to its set of open subsets. If $f : X \to Y$ is continuous, this induces a map $\Omega f : \Omega Y \to \Omega X$. This is representable by the *Sierpiński space* Σ with two points $\{0,1\}$ and open sets

$$\varnothing; \quad \{1\}; \quad \Sigma$$

The continuous maps $f : X \to \Sigma$ are exactly the characteristic functions of the open subsets of X, because continuity is just that $f^{-1}(\{1\})$ is open.

(vi) The dual vector space functor $(-)^\star : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$ is not representable because its codomain is not **Set**, but composing with the forgetful functor makes it representable by the one-dimensional space k.

(vii) Let G be a group. The (unique up to isomorphism) representable functor $G \to \mathbf{Set}$ is the *Cayley representation* of the group; that is, the set G acting on itself by multiplication.

(viii) Let A, B be objects of a locally small category $\mathcal{C}$. Then there is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sending C to the Cartesian product

$$\mathcal{C}(C,A) \times \mathcal{C}(C,B)$$

If this is representable, we call the representing object a categorical *product* of A and B, and denote it $A \times B$. The universal element is a pair of morphisms $\pi_1 : A \times B \to A, \pi_2 : A \times B \to B$, called *projections*. This has the property that for any pair $(f : C \to A, g : C \to B)$ there exists a unique morphism $h = (f,g) : C \to A \times B$ satisfying $\pi_1 h = f, \pi_2 h = g$.

(ix) Dually, there is the notion of a *coproduct* $A + B$, which is a representing object of the functor mapping C to

$$\mathcal{C}(A,C) \times \mathcal{C}(B,C)$$

with *coprojections* $\nu_1 : A \to A + B, \nu_2 : B \to A + B$.

(x) Let $f, g : A \rightrightarrows B$ be a parallel pair of morphisms in a locally small category $\mathcal{C}$. Define a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ by sending C to

$$\{h : C \to A \mid fh = gh\}$$

If this is representable, we call the representation an *equaliser* of f and g. This consists of a representing object E with a morphism $e : E \to A$ satisfying $fe = ge$. Moreover, for any morphism h with $fh = gh$, h factors uniquely through e. Hence, e is a monomorphism. Monomorphisms that occur in this way are called *regular*.

(xi) Dually, there is also a notion of coequaliser, giving rise to an epimorphism. We again call epimorphisms *regular* if they arise in this way.

In **Set**, the categorical product is the Cartesian product, and the categorical coproduct is the disjoint union. The equaliser of $f, g : A \rightrightarrows B$ is the set

$$\{a \in A \mid fa = ga\}$$

The coequaliser of f, g is the quotient

$$B/\sim$$

where $\sim$ is the equivalence relation generated by $fa \sim ga$.

In **Gp**, the product is the direct product, but the coproduct is the *free product* $A * B$. The equaliser of $f, g : A \rightrightarrows B$ is as in **Set**, which is a subgroup of A. The coequaliser of f, g is the quotient by the smallest congruence containing all pairs (fa, ga). In **Set** and **Gp**, all monomorphisms and epimorphisms are regular.

In **Top**, not all injections or surjections are regular monomorphisms or epimorphisms.

2.3 Separating and detecting families

Definition. Let $\mathcal{C}$ be a locally small category, and $\mathcal{G}$ a class of objects of $\mathcal{C}$. We say that

(i) $\mathcal{G}$ is a *separating family* for $\mathcal{C}$ if the functors $\mathcal{C}(G, -)$ for $G \in \mathcal{G}$ are collectively faithful; that is, if $f, g : A \rightrightarrows B$, the equations $fh = gh$ for all $h : G \to A$ with $G \in \mathcal{G}$ imply $f = g$.

$$\begin{tikzcd} G \arrow[r, "h"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

(ii) $\mathcal{G}$ is a *detecting family* for $\mathcal{C}$ if the functors $\mathcal{C}(G, -)$ for $G \in \mathcal{G}$ collectively *reflect isomorphisms*; that is, if $f : A \to B$ is such that every $h : G \to B$ with $G \in \mathcal{G}$ factors uniquely through A, then f is an isomorphism.

$$\begin{tikzcd} G \arrow[r, dashed, "g"] \arrow[dr, "h"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

If $\mathcal{G} = \{G\}$, we call G a *separator* or *detector* respectively.

Separating and detecting families are both sometimes called *generating families*.

Lemma. (i) If $\mathcal{C}$ has equalisers, then any detecting family is separating.
(ii) If $\mathcal{C}$ is balanced, then any separating family is detecting.

Proof. Part (i). Suppose $\mathcal{G}$ is detecting, and $f, g : A \rightrightarrows B$ such that every morphism $h : G \to A$ with $G \in \mathcal{G}$ has $fh = gh$. Then every such $h : G \to A$ with $G \in \mathcal{G}$ factors uniquely through the equaliser of f and g.

$$\begin{tikzcd} G \arrow[d, dashed] \arrow[dr, "h"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Thus this equaliser e must be an isomorphism as $\mathcal{G}$ is detecting. Since $ef = eg$, we must have $f = g$, as required.

Part (ii). Suppose $\mathcal{G}$ is separating, and $f : A \to B$ is such that every $h : G \to B$ with $G \in \mathcal{G}$ factors uniquely through f. As $\mathcal{C}$ is balanced, it suffices to show that f is both monic and epic.

If $fg = fh$ for some $g, h : C \rightrightarrows A$, then any $k : G \to C$ with $G \in \mathcal{G}$ satisfies $gk = hk$, since both are factorisations of $fgk = fhk$ through f.

$$\begin{tikzcd} G \arrow[r, "k"] & C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Since $\mathcal{G}$ is separating, $g = h$. As this is true for all pairs g, h, we must have that f is monic.

Similarly, if $\ell, m : B \rightrightarrows D$ satisfy $\ell f = mf$, then any $n : G \to B$ with $G \in \mathcal{G}$ satisfies $\ell n = mn$, since it factors through f.

$$\begin{tikzcd} & G \arrow[dl, dashed] \arrow[d, "n"] & \\ A \arrow[r, "f"'] & B \arrow[r, shift left, "\ell"] \arrow[r, shift right, "m"'] & D \end{tikzcd}$$

So $\ell = m$, giving that f is epic. □

Example. (i) In **Gp**, the forgetful functor is represented by $\mathbb{Z}$. This functor is faithful and reflects isomorphisms, so it is a separator and a detector.

(ii) In **Rng**, the forgetful functor is represented by $\mathbb{Z}[x]$, so similarly $\mathbb{Z}[x]$ is a separator and a detector.

(iii) If $\mathcal{C}$ is small, the set $\{\mathcal{C}(A, -) \mid A \in \operatorname{ob} \mathcal{C}\}$ is a separating and detecting set for $[\mathcal{C}, \mathbf{Set}]$ by the Yoneda lemma.

(iv) In **Top**, the one-point space 1 is a separator, but **Top** has no detecting set. If κ is an infinite cardinal, let X_κ be a discrete space of cardinality κ, and let Y_κ be the same set with the co-$< \kappa$ topology:

$$U \text{ open} \iff U = \varnothing \text{ or } |Y_\kappa \setminus U| < \kappa$$

The identity $X_\kappa \to Y_\kappa$ is continuous but not a homeomorphism. Given any set $\mathcal{G}$ of spaces, if κ is larger than $|G|$ for all $G \in \mathcal{G}$, then $\mathcal{G}$ cannot detect the fact that the map $X_\kappa \to Y_\kappa$ is not a homeomorphism.

(v) Let $\mathcal{C}$ be the category whose objects are the (von Neumann) ordinals, and in addition to the identity morphisms, there are precisely two morphisms $f, g : \alpha \rightrightarrows \beta$ when $\alpha < \beta$. We define composition in such a way that $ff = fg = gf = gg = f$. Now, 0 is a detector for $\mathcal{C}$: it detects that $f, g : 0 \rightrightarrows \alpha$ are not isomorphisms, as neither factors through the other, and it detects that $f, g : \alpha \rightrightarrows \beta$ are not isomorphisms for $0 < \alpha < \beta$ since the morphism $g : 0 \to \beta$ does not factor through either of them. There is no separating set for $\mathcal{C}$: for any set of ordinals $\mathcal{G}$, if $\alpha > \gamma$ for all $\gamma \in \mathcal{G}$, $\mathcal{G}$ cannot separate $f, g : \alpha \rightrightarrows \alpha + 1$.

(vi) **Gp** has no *coseparating* or *codetecting* set of objects. Given any set $\mathcal{G}$ of groups, let H be a simple group with cardinality greater than that of each element of $\mathcal{G}$. Then the only homomorphisms from H to elements of $\mathcal{G}$ are trivial. In particular, $\mathcal{G}$ cannot detect that the map $H \to 1$ is not an isomorphism.

2.4 Projectivity

The functors $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ preserve monomorphisms. They do not, in general, preserve epimorphisms.

Definition. We say that an object P of a locally small category $\mathcal{C}$ is *projective* if $\mathcal{C}(P, -)$ preserves epimorphisms. In more elementary terms, given a diagram

$$\begin{tikzcd} & P \arrow[d, "f"] \\ Q \arrow[r, two heads, "g"'] & R \end{tikzcd}$$

there exists $h : P \to Q$ such that $gh = f$.

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "h"'] \\ Q \arrow[r, two heads, "g"'] & R \end{tikzcd}$$

If this holds for all g in some class $\mathcal{E}$ of epimorphisms, we say that P is $\mathcal{E}$-projective. The dual notion is called *injectivity*.

We will consider the class of pointwise epimorphisms in $[\mathcal{C}, \mathbf{Set}]$; that is, those natural transformations α whose components α_A are surjective.

Corollary. Objects of the form $\mathcal{C}(A, -)$ are pointwise projective in $[\mathcal{C}, \mathbf{Set}]$.

Proof. If $P = \mathcal{C}(A, -)$, an f in the above diagram corresponds to some $\Phi(f) \in RA$ by the Yoneda lemma. But g_A is surjective, so there exists $\Phi(h) \in QA$ mapping to $\Phi(f)$. □

Proposition. If $\mathcal{C}$ is small, then $[\mathcal{C}, \mathbf{Set}]$ has *enough pointwise projectives*; that is, for any object F there exists a pointwise epimorphism $P \to F$ with P pointwise projective.

Proof. Let $P = \coprod_{(A,x)} \mathcal{C}(A,-)$ where the disjoint union is taken over all pairs (A,x) with $A \in \operatorname{ob}\mathcal{C}$ and $x \in FA$. Then P is pointwise projective, since the $\mathcal{C}(A,-)$ are. There is a natural transformation $\alpha : P \to F$ where the (A,x)-indexed term is $\Psi(x) : \mathcal{C}(A,-) \to F$. This is pointwise epic, since any $x \in FA$ is in the image of $\Psi(x)$. □

3 Adjunctions

3.1 Definition and examples

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, together with a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and $A \to HB$ in $\mathcal{C}$, which is natural in both variables A, B. We say that F is the *left adjoint* to G, and that G is the *right adjoint* to F, and write $F \dashv G$.

If $\mathcal{C}, \mathcal{D}$ are locally small, then the naturality condition is that

$$\mathcal{D}(F-,-); \quad \mathcal{C}(-,G-)$$

are naturally isomorphic functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$.

Example. (i) The free group functor $F : \mathbf{Set} \to \mathbf{Gp}$ is left adjoint to the forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$.

$$\mathbf{Gp}(FA, G) \leftrightarrow \mathbf{Set}(A, UG)$$

(ii) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint $D : \mathbf{Set} \to \mathbf{Top}$ which equips each set with its discrete topology.

$$\mathbf{Top}(DX, Y) \leftrightarrow \mathbf{Set}(X, UY)$$

It also has a right adjoint $I : \mathbf{Set} \to \mathbf{Top}$ which equips each set with its indiscrete topology.

$$\mathbf{Set}(UX, Y) \leftrightarrow \mathbf{Top}(X, IY)$$

(iii) Consider the functor $\operatorname{ob} : \mathbf{Cat} \to \mathbf{Set}$ which maps each category to its set of objects. It has a left adjoint D which turns each set X into a discrete category in which the objects are elements of X, and the only morphisms are identities. It also has a right adjoint I which turns each set X into an indiscrete category in which the objects are elements of X, and there is exactly one morphism between any two elements of X. In addition, $D : \mathbf{Set} \to \mathbf{Cat}$ has a left adjoint $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$, where $\pi_0\mathcal{C}$ is the set of connected components of $\operatorname{ob}\mathcal{C}$ under the graph induced by its morphisms.

$$\mathbf{Set}(\pi_0\mathcal{C}, X) \leftrightarrow \mathbf{Cat}(\mathcal{C}, DX); \quad \mathbf{Cat}(DX, \mathcal{C}) \leftrightarrow \mathbf{Set}(X, \operatorname{ob}\mathcal{C}); \quad \mathbf{Set}(\operatorname{ob}\mathcal{C}, X) \leftrightarrow \mathbf{Cat}(\mathcal{C}, IX)$$

Thus we have a chain

$$\pi_0 \dashv D \dashv \operatorname{ob} \dashv I$$

(iv) For any set A, we have a functor $(-) \times A : \mathbf{Set} \to \mathbf{Set}$. This functor has a right adjoint, which is the functor $\mathbf{Set}(A,-) : \mathbf{Set} \to \mathbf{Set}$.

$$\mathbf{Set}(B \times A, C) \leftrightarrow \mathbf{Set}(B, \mathbf{Set}(A, C))$$

Applying this bijection is sometimes called *currying* or *λ-conversion*. We say that a category $\mathcal{C}$ with binary products is *cartesian closed* if $(-) \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint, written $[A, -]$ or $(-)^A$, for each A. For example, **Cat** is cartesian closed, where $\mathcal{D}^{\mathcal{C}} = [\mathcal{C}, \mathcal{D}]$ is the functor category that this notation already refers to.

(v) An equivalence $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ forms adjunctions both ways: $F \dashv G, G \dashv F$.

(vi) Let **Idem** be the category of pairs (A, e) where A is a set and e is an idempotent endomorphism $A \to A$. The morphisms in **Idem** are the maps of sets which commute with the idempotents. We have a functor $F : \textbf{Set} \to \textbf{Idem}$ sending A to $(A, 1_A)$. Consider $G : \textbf{Idem} \to \textbf{Set}$ sending (A, e) to the set of fixed points of e. Then $F \dashv G$ since any morphism $FA \to (B, e)$ takes values in $G(B, e)$. But also $G \dashv F$, since a morphism $(A, e) \to FB$ is entirely determined by its action on the fixed points in A under e, because $f(a) = f(ea)$. This is not an equivalence of categories, because G is not faithful. So not all pairs of functors that are adjoint in both directions form an equivalence.

(vii) Let $\mathcal{C}$ be a category. There is a unique functor $G : \mathcal{C} \to \mathbf{1}$, where $\mathbf{1}$ is the discrete category on a single object. A left adjoint for G, if it exists, sends the object in $\mathbf{1}$ to an *initial object* I of $\mathcal{C}$, which is an object with a unique morphism to every object in $\mathcal{C}$. Dually, a right adjoint sends the object in $\mathbf{1}$ to a *terminal object* T, which is an object with a unique morphism from every object in $\mathcal{C}$. In **Set**, the empty set is initial, and any singleton is terminal. In **Gp**, the trivial group is initial and terminal.

(viii) Let $f : A \to B$ be a function of sets, and let $A' \subseteq A, B' \subseteq B$. Then $Pf(A') \subseteq B'$ if and only if $A' \subseteq P^\star f(B')$. Thus $Pf \dashv P^\star f$ as functors between PA and PB as posets.

(ix) Let A, B be sets with a relation $R \subseteq A \times B$. We define mappings $(-)^r : PA \to PB$ by

$$S^r = \{b \in B \mid \forall a \in S,\ (a, b) \in R\}$$

and $(-)^\ell : PB \to PA$ by

$$T^\ell = \{a \in A \mid \forall b \in T,\ (a, b) \in R\}$$

These are contravariant functors, and

$$S \subseteq T^\ell \iff S \times T \subseteq R \iff T \subseteq S^r$$

We say that $(-)^\ell$ and $(-)^r$ are *adjoint on the right*. This pair is called a *Galois connection*.

(x) The contravariant power-set functor $P^\star$ is self-adjoint on the right, since functions $A \to P^\star B$ and $B \to P^\star A$ naturally correspond bijectively to subsets of $A \times B$.

(xi) The dual vector space functor $(-)^\star : \textbf{Vect}_k \to \textbf{Vect}_k$ is self-adjoint on the right, as linear maps $V \to W^\star$ and linear maps $W \to V^\star$ both naturally correspond to bilinear forms on $V \times W$.

3.2 Comma categories

Definition. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and $A \in \operatorname{ob} \mathcal{C}$. Then, the *comma category* $(A \downarrow G)$ is the category whose objects are pairs (B, f) where $B \in \operatorname{ob} \mathcal{D}$ and $f : A \to GB$ in $\mathcal{C}$, and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ which commute with f, f':

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

Theorem. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then specifying a left adjoint for G is equivalent to specifying an initial object of the comma categories $(A \downarrow G)$ for each A.

Proof. First, note that an object (B, f) is initial in $(A \downarrow G)$ if and only if for every (B', f'), there is a unique morphism $g : B \to B'$ such that the following triangle commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & GB \arrow[d, "Gg"] \\ & GB' \end{tikzcd}$$

Suppose $F \dashv G$. Then let $\eta_A : A \to GFA$ correspond to the identity 1_{FA} under the adjunction. We show that (FA, η_A) is initial in $(A \downarrow G)$. Indeed, given $f : A \to GB$, then

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GFA \arrow[d, "Gg"] \\ & GB \end{tikzcd}$$

commutes if and only if g is the morphism corresponding to f under the adjunction. In particular, for any f, there is a unique such g.

Conversely, suppose (FA, η_A) is initial in $(A \downarrow G)$ for each A. Then we define the action of F on objects by mapping A to FA. We make F into a functor by mapping $f : A \to A'$ to the unique morphism that makes the following square commute; this exists as (FA, η_A) is initial.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & GFA \arrow[d, "GFf"] \\ A' \arrow[r, "\eta_{A'}"'] & GFA' \end{tikzcd}$$

Functoriality of F follows from the uniqueness of Ff. The bijection between morphisms $f : A \to GB$ and $g : FA \to B$ sends f to the unique g giving $(Gg)\eta_A = f$. Naturality of the bijection in A was built in to the definition of F as a functor, and naturality in B is easy. $\square$

Corollary. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be left adjoints to $G : \mathcal{D} \to \mathcal{C}$. Then $F \simeq F'$ in $[\mathcal{C}, \mathcal{D}]$.

Proof. (FA, η_A) and $(F'A, \eta'_A)$ are both initial objects in $(A \downarrow G)$, and so there is a unique isomorphism $\alpha_A : (FA, \eta_A) \to (F'A, \eta'_A)$ in this category. The map $A \mapsto \alpha_A$ is natural, because given $f : A \to A'$, $\alpha_{A'}(Ff)$ and $(F'f)\alpha_A$ are both morphisms $(FA, \eta_A) \rightrightarrows (F'A', \eta'_{A'} f)$ from an initial object in $(A \downarrow G)$, so must be equal. $\square$

Lemma. Suppose

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "H"] & \mathcal{E} \arrow[l, shift left, "K"] \end{tikzcd}$$

where $F \dashv G$ and $H \dashv K$. Then $HF \dashv GK$.

Proof. We have bijections

$$\mathcal{E}(HFA, C) \leftrightarrow \mathcal{D}(FA, KC) \leftrightarrow \mathcal{C}(A, GKC)$$

which are natural in A and C, so their composite is also natural. □

Corollary. Suppose the square of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[d, "H"] \\ \mathcal{E} \arrow[r, "K"'] & \mathcal{F} \end{tikzcd}$$

commutes, and all of the functors F, G, H, K have left adjoints F', G', H', K'. Then the square of left adjoints

$$\begin{tikzcd} \mathcal{C} & \mathcal{D} \arrow[l, "F'"'] \\ \mathcal{E} \arrow[u, "G'"] & \mathcal{F} \arrow[l, "K'"] \arrow[u, "H'"'] \end{tikzcd}$$

commutes up to natural isomorphism.

This result holds for any shape of diagram, not just a square. The hypothesis can be weakened to only require that the first diagram commutes up to natural isomorphism.

Proof. The two composites $F'H'$ and $G'K'$ are left adjoints to $HF = KG$, so must be naturally isomorphic. □

3.3 Units and counits

Given an adjunction $F \dashv G$, the proof of the previous theorem demonstrated a naturality square between the morphisms $\eta_A : A \to GFA$ corresponding to 1_{FA} under the adjunction. We call $\eta : 1_{\mathcal{C}} \to GF$ the *unit* of the adjunction. Dually, the map $\epsilon : FG \to 1_{\mathcal{D}}$ is called the *counit* of the adjunction; each $\epsilon_B : FGB \to B$ corresponds to 1_{GB}.

Theorem. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$. Specifying an adjunction $F \dashv G$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$, $\epsilon : FG \to 1_{\mathcal{D}}$, satisfying the *triangular* *identities*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Proof. Suppose we have an adjunction $F \dashv G$. We have seen how to define η and ϵ; it thus suffices to check the triangular identities. Since they are dual to each other, it suffices to check the first. The morphism ϵ_{FA} corresponds under the adjunction to 1_{GFA}, so by naturality, the composite $\epsilon_{FA}(F\eta_A)$ corresponds to $1_{GFA}\eta_A = \eta_A$. But 1_{FA} corresponds to η_A, giving the commutative triangle $\epsilon_{FA}(F\eta_A) = 1_{FA}$.

Conversely, suppose η and ϵ are natural transformations satisfying the triangular identities. We map $f : A \to GB$ to the composite $\Phi(f)$ given by

$$FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$$

and $g : FA \to B$ to the composite $\Psi(g)$ given by

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$$

These assignments are natural in A and B as η and ϵ are natural transformations. Thus it suffices to show $\Psi\Phi$ and $\Phi\Psi$ are the relevant identity maps; again they are dual so it suffices to show $\Psi\Phi(f) = f$. $\Psi\Phi(f)$ is the composite

$$A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB$$

which by naturality of η is equal to

$$A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB$$

which is equal to f by the triangular identity. □

Recall that an equivalence of categories consisted of isomorphisms $\alpha : 1_{\mathcal{C}} \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. These isomorphisms may not satisfy the triangular identities, but we can always choose α and β in such a way that these identities hold.

Proposition. Let (F, G, α, β) be an equivalence of categories. Then there exist natural isomorphisms $\alpha' : 1_{\mathcal{C}} \to GF$ and $\beta' : FG \to 1_{\mathcal{D}}$ which satisfy the triangular identities. In particular, $F \dashv G \dashv F$.

Proof. We will set $\alpha' = \alpha$, and construct β' to be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$$

Note that $FG\beta = \beta_{FG}$, since

$$\begin{tikzcd} FGFG \arrow[r, "FG\beta"] \arrow[d, "\beta_{FG}"'] & FG \arrow[d, "\beta"] \\ FG \arrow[r, "\beta"'] & 1_{\mathcal{D}} \end{tikzcd}$$

commutes by naturality of β. Note also that β is monic. Dually, note that $GF\alpha = \alpha_{GF}$. For the triangular identities, consider the diagrams

$$\begin{tikzcd}
F \arrow[r, "F\alpha"] \arrow[dr, "1_F"'] & FGF \arrow[r, "(\beta_{FGF})^{-1}"] \arrow[d, "(F\alpha)^{-1}"] & FGFGF \arrow[d, "(F\alpha_{GF})^{-1}=(FGF\alpha)^{-1}"] \\
& F \arrow[r, "\beta_F"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\beta_F"] \\
& & F
\end{tikzcd}$$

and

$$\begin{tikzcd}
G \arrow[r, "\alpha_G"] \arrow[dr, "1_G"'] & GFG \arrow[r, "(GFG\beta)^{-1}"] \arrow[d, "\alpha_G^{-1}"] & GFGFG \arrow[d, "(GF\alpha_G)^{-1}=(\alpha_{GFG})^{-1}"] \\
& G \arrow[r, "(G\beta)^{-1}"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\beta"] \\
& & G
\end{tikzcd}$$

where the squares commute by naturality of β and α respectively. Thus α', β' are the unit and counit of an adjunction $F \dashv G$ as required. Similarly, $(\beta')^{-1}, (\alpha')^{-1}$ are the unit and counit of an adjunction $G \dashv F$. □

Lemma. Let $F \dashv G$ be an adjunction with counit $\epsilon : FG \to 1_{\mathcal{D}}$. Then
(i) ϵ is pointwise epimorphic if and only if G is faithful;
(ii) ϵ is a (pointwise) isomorphism if and only if G is full and faithful.

Proof. Part (i). Given $g : B \to B'$ in $\mathcal{D}$, the composite $g\epsilon_B$ corresponds under the adjunction to $Gg : GB \to GB'$. Thus for morphisms g with specified domain and codomain, the map $g \mapsto g\epsilon_B$ is injective if and only if the action of G is injective. This is true for all B and B' if and only if ϵ is pointwise epimorphic, if and only if G is faithful.

Part (ii). Similarly, G is full and faithful if and only if the map $g \mapsto g\epsilon_B$ is a bijection on morphisms with specified domain and codomain. This clearly holds if ϵ_B is an isomorphism for all B. Conversely, if the condition holds, there is a unique map $g : B \to FGB$ such that $\epsilon_B g = 1_B$. Then $\epsilon_B g \epsilon_B = \epsilon_B$, so $g\epsilon_B$ and 1_{FGB} have the same composite with ϵ_B, so they are equal. □

3.4 Reflections

Definition. An adjunction $F \dashv G$ is called a *reflection* if the counit is an isomorphism. Dually, it is called a *coreflection* if the unit is an isomorphism. A full subcategory is called *reflective* if the inclusion functor has a left adjoint; in this case the adjunction is a reflection.

Remark. If $F \dashv G$ is a reflection, then $G : \mathcal{D} \to \mathcal{C}$ induces an equivalence of categories between $\mathcal{D}$ and the full subcategory of $\mathcal{C}$ on the objects in the image of G. This subcategory is reflective.

If $\mathcal{D} \subseteq \mathcal{C}$ is a reflective subcategory, there is intuitively a best possible way to get *into* $\mathcal{D}$ from some object in $\mathcal{C}$. The left adjoint sends an object in $\mathcal{C}$ to its 'best approximation' in $\mathcal{D}$. If $\mathcal{D}$ is coreflective, there is a best possible way to get *out of* $\mathcal{D}$ to some object in $\mathcal{C}$.

Example. (i) **AbGp** is reflective in **Gp**; the left adjoint to the inclusion map sends a group G to its abelianisation $G^{\mathrm{ab}} = {}^{G}\!/_{H}$, the quotient of G by its commutator subgroup $H = \{aba^{-1}b^{-1} \mid a, b \in G\} \trianglelefteq G$. Note that any homomorphism $G \to A$ where A is abelian factors uniquely through the quotient map $G \to G^{\mathrm{ab}}$, giving the adjunction as required.

(ii) Recall that an abelian group is called *torsion* if all of its elements have finite order, and *torsion-free* if all of its nonzero elements have infinite order. For an abelian group A, its set of torsion elements forms a subgroup A_t, which is a torsion group. Any homomorphism from a torsion group to A must factor through A_t. Thus A_t is the coreflection of A in the category of torsion abelian groups, and ${}^{A}\!/_{A_t}$ is the reflection of A in the category of torsion-free abelian groups.

(iii) The full subcategory **KHaus** of compact Hausdorff spaces is reflective in the category **Top** of topological spaces. The left adjoint to the inclusion map is the *Stone–Čech compactification* functor β. We will construct this functor using the special adjoint functor theorem, which is explored in the next section.

(iv) Recall that a subset C of a topological space X is called *sequentially closed* if for every sequence $x_n \in C$ converging to a limit $x \in X$, we have $x \in C$. We say that X is a *sequential space* if all sequentially closed subsets are closed. The full subcategory **Seq** of sequential spaces is coreflective in **Top**. Given a space X, let X_s denote the same set, but where the topology is such that all sequentially closed sets are also taken to be closed. The identity map $X_s \to X$ is continuous, and forms the counit of the adjunction.

(v) The category **Preord** of preorders is reflective in **Cat**. The left adjoint maps a category $\mathcal{C}$ to the quotient category ${}^{\mathcal{C}}\!/_{\sim}$ where $\sim$ identifies all parallel pairs of morphisms.

(vi) Let X be a topological space. Then the poset ΩX of open sets in X is coreflective in the poset PX, since if U is open and A is an arbitrary subset of X, then $U \subseteq A$ if and only if $U \subseteq A^\circ$. Thus the interior operator $(-)^\circ$ is right adjoint to the inclusion $\Omega X \to PX$. Dually, the poset of closed sets is reflective in PX; the closure operator $\overline{(-)}$ is left adjoint to the inclusion.

4 Limits

4.1 Cones over diagrams

To formally define limits and colimits, we first need to define more precisely what is meant by a diagram in a category.

Definition. Let J be a category, which will almost always be small, and often finite. A *diagram* of shape J in a category $\mathcal{C}$ is a functor $D : J \to \mathcal{C}$.

We call the objects $D(j)$ the *vertices* of the diagram, and the morphisms $D(\alpha)$ the *edges* of the diagram.

Example. Let J be the finite category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

A diagram of shape J in $\mathcal{C}$ is exactly a commutative square in $\mathcal{C}$. The diagonal arrow is required to make J into a category.

Example. Let J be the finite category

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Then a diagram of shape J in $\mathcal{C}$ is a square of objects in $\mathcal{C}$ whose morphisms may or may not commute.

Definition. Let D be a diagram of shape J in $\mathcal{C}$. A *cone over* D consists of an object $A \in \operatorname{ob} \mathcal{C}$ called the *apex* of the cone, together with morphisms $\lambda_j : A \to D(j)$ called the *legs* of the cone, such that all triangles of the following form commute.

$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"'] & & D(j') \end{tikzcd}$$

We can define the notion of a morphism between cones.

Definition. Let $(A, \lambda_j), (B, \mu_j)$ be cones over a diagram D of shape J in $\mathcal{C}$. Then a *morphism of cones* is a morphism $f : A \to B$ such that all triangles of the following form commute.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\lambda_j"'] & & B \arrow[dl, "\mu_j"] \\ & D(j) & \end{tikzcd}$$

This makes the class of cones over a diagram D into a category, which will be denoted $\mathrm{Cone}(D)$.

Remark. A cone over a diagram D with apex A is the same as a natural transformation from the constant diagram ΔA to D, as we can expand the commutative triangles into the following form.

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "\lambda_j"'] & A \arrow[d, "\lambda_{j'}"] \\ D(j) \arrow[r, "D(\alpha)"'] & D(j') \end{tikzcd}$$

Note that Δ is a functor $\mathcal{C} \to [J, \mathcal{C}]$, and thus $\mathrm{Cone}(D)$ is exactly the comma category $(\Delta \downarrow D)$.

4.2 Limits

Definition. A *limit* for a diagram D of shape J in $\mathcal{C}$ is a terminal object in the category of cones over D. Dually, a *colimit* for D is an initial object in the category of cones under D.

A cone under a diagram is often called a *cocone*.

Remark. Using the fact that $\mathrm{Cone}(D) = (\Delta \downarrow D)$ where $\Delta : \mathcal{C} \to [J, \mathcal{C}]$, the category $\mathcal{C}$ has limits for all diagrams of shape J if and only if Δ has a right adjoint.

Example. (i) If J is the empty category, there is a unique diagram D of shape J in any category $\mathcal{C}$. Thus, a cone over this diagram is just an object in $\mathcal{C}$, and morphisms of cones are just morphisms in $\mathcal{C}$. In particular, $\mathrm{Cone}(D) \cong \mathcal{C}$, so a limit for D is a terminal object in $\mathcal{C}$. Dually, a colimit of the empty diagram is an initial object.

(ii) Let J be the discrete category with two objects. A diagram of shape J in $\mathcal{C}$ is thus a pair of objects. A cone over this diagram is a *span*.

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

A limit cone is precisely a categorical product $A \times B$.

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A & & B \end{tikzcd}$$

Similarly, the colimit for a pair of objects is a categorical coproduct $A + B$.

(iii) If J is any discrete category, a diagram of shape J is a family of objects A_j in $\mathcal{C}$ indexed by the objects of J. Limits and colimits over this diagram are products and coproducts of the A_j.

(iv) If J is the category $\bullet \rightrightarrows \bullet$, a diagram of shape J is a parallel pair of morphisms $f, g : A \rightrightarrows B$. A cone over such a parallel pair is

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[dr, "k"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

satisfying $fh = k = gh$. Equivalently, it is a morphism $h : C \to A$ satisfying $fh = gh$. Thus, a limit is an equaliser, and dually, a colimit is a coequaliser.

(v) Let J be the category

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

A diagram of shape J is thus a cospan in $\mathcal{C}$.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

A cone over this diagram is

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] \arrow[dr, "\ell"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

where $\ell = fh = gk$ is redundant. Thus a cone is a span that completes the commutative square. A limit for the cospan is the universal way to complete this commutative square, which is called a *pullback* of f and g. Dually, colimits of spans are called *pushouts*.

If any category $\mathcal{C}$ has binary products and equalisers, we can construct all pullbacks. First, we construct the product $A \times B$, then we form the equaliser of $f\pi_1, g\pi_2 : A \times B \rightrightarrows C$. This yields the pullback.

(vi) Let M be the two-element monoid $\{1, e\}$ with $e^2 = e$. A diagram of shape M in a category $\mathcal{C}$ is an object of $\mathcal{C}$ equipped with an idempotent endomorphism. A cone over this diagram is a morphism $f : B \to A$ such that $ef = f$. A limit (respectively colimit) is the monic (respectively epic) part of a splitting of e. This is because the pair $(e, 1_A)$ has an equaliser if and only if e splits.

(vii) Let $\mathbb{N}$ be the poset category of the natural numbers. A diagram of shape $\mathbb{N}$ is a *direct sequence* of objects, which consists of objects $A_0, A_1, \dots$ and morphisms $f_i : A_i \to A_{i+1}$. A colimit for this diagram is a *direct limit*, which consists of an object A_∞ and morphisms $g_i : A_i \to A_\infty$ which are compatible with the f_i. Dually, an *inverse sequence* is a diagram of shape $\mathbb{N}^{\mathrm{op}}$, and a limit for this diagram is called an *inverse limit*. For example, an infinite-dimensional CW-complex X is the direct limit of its n-dimensional skeletons in **Top**. The ring of p-adic integers is the limit of the inverse sequence defined by $A_n = \mathbb{Z}/p^n\mathbb{Z}$ in **Rng**.

Lemma. Let $\mathcal{C}$ be a category.
(i) If $\mathcal{C}$ has equalisers and all small products, then $\mathcal{C}$ has all small limits.
(ii) If $\mathcal{C}$ has equalisers and all finite products, then $\mathcal{C}$ has all finite limits.
(iii) If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.

Note that the empty product is implicitly included in (i) and (ii). A terminal object is a product over no factors.

Proof. Parts (i) and (ii). We prove (i) and (ii) in the same way. We will first construct the product P of the $D(j)$ for each $j \in \operatorname{ob} J$. Then, we will use an equaliser to construct the subobject E of P that simultaneously satisfies all of the equations required for E to be the apex of a cone. The fact that we have used an equaliser will show that this is a limit cone.

Let $D : J \to \mathcal{C}$ be a diagram. We form the products

$$P = \prod_{j \in \operatorname{ob} J} D(j); \quad Q = \prod_{\alpha \in \operatorname{mor} J} D(\operatorname{cod} \alpha)$$

These are small or finite as required. Using the universal property of the product on Q, we have morphisms $f, g : P \rightrightarrows Q$ defined by

$$\pi_\alpha f = \pi_{\operatorname{cod} \alpha} : P \to D(\operatorname{cod} \alpha); \quad \pi_\alpha g = D(\alpha)\pi_{\operatorname{dom} \alpha} : P \to D(\operatorname{cod} \alpha)$$

For $\alpha : j \to j'$ in D, these morphisms are represented by

$$\begin{tikzcd} P \arrow[r, dashed, "f"] \arrow[dr, "\pi_{j'}"'] & Q \arrow[d, "\pi_\alpha"] \\ & D(j') \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, dashed, "g"] \arrow[d, "\pi_j"'] & Q \arrow[d, "\pi_\alpha"] \\ D(j) \arrow[r, "D(\alpha)"'] & D(j') \end{tikzcd}$$

Let $e : E \to P$ be an equaliser for f and g, and define $\lambda_j = \pi_j e : E \to D(j)$. Then for each $\alpha : j \to j'$, the following diagram commutes.

$$\begin{tikzcd} & E \arrow[d, "e"] \arrow[ddl, bend right, "\lambda_j"'] \arrow[ddr, bend left, "\lambda_{j'}"] & \\ & P \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"'] & & D(j') \end{tikzcd}$$

Therefore, these morphisms form a cone. Given any cone $(A, (\mu_j)_{j \in \operatorname{ob} J})$ over D, we have a unique $\mu : A \to P$ with $\pi_j \mu = \mu_j$ for all j. Then,

$$\pi_\alpha f \mu = \mu_{\operatorname{cod} \alpha} = D(\alpha)\mu_{\operatorname{dom} \alpha} = \pi_\alpha g \mu$$

for all α, so μ factors uniquely through e.

Part (iii). We show that the hypotheses of (iii) imply those of (ii). If 1 is the terminal object, we form the pullback of the span

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & 1 \end{tikzcd}$$

This has the universal property of the product $A \times B$, so $\mathcal{C}$ has binary products and hence all finite products by induction. To construct the equaliser of $f, g : A \rightrightarrows B$, we consider the pullback of

$$\begin{tikzcd} & A \arrow[d, "(1_A{,}f)"] \\ A \arrow[r, "(1_A{,}g)"'] & A \times B \end{tikzcd}$$

Any cone over this diagram has its two legs $C \rightrightarrows A$ equal, so a pullback is an equaliser for f, g. $\square$

Definition. A category is called *complete* if it has all small limits, and *cocomplete* if it has all small colimits.

Example. The categories **Set**, **Gp**, **Top** are complete and cocomplete.

4.3 Preservation and creation

Definition. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. We say that G

(i) *preserves* limits of shape J if whenever $D : J \to \mathcal{D}$ is a diagram with limit cone $(L, (\lambda_j)_{j \in \operatorname{ob} J})$, the cone $(GL, (G\lambda_j)_{j \in \operatorname{ob} J})$ is a limit for GD;

(ii) *reflects* limits of shape J if whenever $D : J \to \mathcal{D}$ is a diagram and $(L, (\lambda_j)_{j \in \operatorname{ob} J})$ is a cone such that $(GL, (G\lambda_j)_{j \in \operatorname{ob} J})$ is a limit for GD, then $(L, (\lambda_j)_{j \in \operatorname{ob} J})$ is a limit for D;

(iii) *creates* limits of shape J if whenever $D : J \to \mathcal{D}$ is a diagram with limit cone $(M, (\mu_j)_{j \in \operatorname{ob} J})$ for GD in $\mathcal{C}$, there exists a cone $(L, (\lambda_j)_{j \in \operatorname{ob} J})$ over D such that

$(GL, (G\lambda_j)_{j \in \mathrm{ob}\, J}) \cong (M, (\mu_j)_{j \in \mathrm{ob}\, J})$ in Cone(GD), and any such cone is a limit for D.

We typically assume in (i) that $\mathcal{D}$ has all limits of shape J, and we assume in (ii) and (iii) that $\mathcal{C}$ has all limits of shape J. With these assumptions, G creates limits of shape J if and only if G preserves and reflects limits, and $\mathcal{D}$ has all limits of shape J.

Corollary. In any of the statements of the previous lemma, we can replace both instances of '$\mathcal{C}$ has' by either '$\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves' or '$\mathcal{C}$ has and $G : \mathcal{D} \to \mathcal{C}$ creates'.

Example. (i) The forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$ creates all small limits. It does not preserve colimits, as in particular it does not preserve coproducts.

(ii) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ preserves all small limits and colimits, but does not reflect them, as we can retopologise the apex of a limit cone.

(iii) The inclusion $\mathbf{AbGp} \to \mathbf{Gp}$ reflects coproducts, but does not preserve them. A free product of two groups G, H is always nonabelian, except for the case where either G or H is the trivial group, but the coproduct of the trivial group with H is isomorphic to H in both categories.

Lemma. Suppose $\mathcal{D}$ has limits of shape J. Then, for any $\mathcal{C}$, the functor category $[\mathcal{C}, \mathcal{D}]$ also has limits of shape J, and the forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\mathrm{ob}\, \mathcal{C}}$ creates them.

Proof. Given a diagram $D : J \to [\mathcal{C}, \mathcal{D}]$, we can regard it as a functor $D : J \times \mathcal{C} \to \mathcal{D}$, so for a fixed object in $\mathcal{C}$, we obtain a diagram $D(-, A)$ of shape J in $\mathcal{D}$, which has a limit $(LA, (\lambda_{j,A})_{j \in \mathrm{ob}\, J})$. Given any $f : A \to B$ in $\mathcal{C}$, the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, and so factor uniquely through its limit LB. Thus we obtain $Lf : LA \to LB$. This is functorial because Lf is unique with this property. This is the unique lifting of $(LA)_{A \in \mathrm{ob}\, \mathcal{C}}$ to an object of $[\mathcal{C}, \mathcal{D}]$ which makes the $\lambda_{j,-}$ into natural transformations. It is a limit cone in $[\mathcal{C}, \mathcal{D}]$: given any cone in $[\mathcal{C}, \mathcal{D}]$ with apex M and legs $(\mu_{j,-})_{j \in \mathrm{ob}\, J}$ over D, the $\mu_{j,A}$ form a cone over $D(-, A)$, so we obtain a unique $\nu_A : MA \to LA$ such that $\lambda_{j,A} \nu_A = \mu_{j,A}$ for all A. The ν_A form a natural transformation $M \to L$, because for any $f : A \to B$ in $\mathcal{C}$, the two paths $\nu_B(Mf), (Lf)\nu_A : MA \rightrightarrows LB$ are factorisations of the same cone over $D(-, B)$ through its limit, so must be equal. $\square$

Remark. Note that $f : A \to B$ is monic if and only if

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square. Thus, if $\mathcal{D}$ has pullbacks, any monomorphism in $[\mathcal{C}, \mathcal{D}]$ is a pointwise monomorphism, because the pullback in $[\mathcal{C}, \mathcal{D}]$ is constructed pointwise by the previous lemma.

4.4 Interaction with adjunctions

Lemma. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor with a left adjoint. Then G preserves all limits which exist in $\mathcal{D}$.

Proof 1. In this proof, we will assume that $\mathcal{C}, \mathcal{D}$ both have all limits of shape J. If $F \dashv G$, then the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\Delta"'] & \mathcal{D} \arrow[d, "\Delta"] \\ {[J,\mathcal{C}]} \arrow[r, "{[J,F]}"'] & {[J,\mathcal{D}]} \end{tikzcd}$$

commutes. All of the functors in this diagram have right adjoints, so the diagram

$$\begin{tikzcd} \mathcal{C} & \mathcal{D} \arrow[l, "G"'] \\ {[J,\mathcal{C}]} \arrow[u, "\lim_J"] & {[J,\mathcal{D}]} \arrow[l, "{[J,G]}"] \arrow[u, "\lim_J"'] \end{tikzcd}$$

commutes up to natural isomorphism, where $\lim_J$ sends a diagram of shape J to the apex of its limit cone. But this is exactly the statement that G preserves limits. □

Proof 2. In this proof, we will not assume that $\mathcal{C}$ has limits of any kind, and only assume a single diagram $D : J \to \mathcal{D}$ has a limit cone $(L, (\lambda_j)_{j \in \mathrm{ob}\, J})$ over it. Given any cone over GD with apex A and legs $\mu_j : A \to GD(j)$, the legs correspond under the adjunction to morphisms $\overline{\mu}_j : FA \to D(j)$, which form a cone over D by naturality of the adjunction. We obtain a unique factorisation $\overline{\mu} : FA \to L$ with $\lambda_j \overline{\mu} = \overline{\mu}_j$ for all j, or equivalently, $(G\lambda_j)\mu = \mu_j$, where $\mu : A \to GL$ corresponds to $\overline{\mu}$ under the adjunction. □

Suppose that $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits. The adjoint functor theorems say that G has a left adjoint, under various assumptions.

Lemma. Suppose that $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then for any $A \in \mathrm{ob}\, \mathcal{C}$, the category $(A \downarrow G)$ has limits of shape J, and the forgetful functor $U : (A \downarrow G) \to \mathcal{D}$ creates them.

Proof. Let $D : J \to (A \downarrow G)$ be a diagram. We write each $D(j)$ as $(UD(j), f_j)$ where $f_j : A \to GUD(j)$. Let $(L, (\lambda_j)_{j \in \mathrm{ob}\, J})$ be a limit for UD in $\mathcal{D}$. By assumption, $(GL, (G\lambda_j)_{j \in \mathrm{ob}\, J})$ is a limit for GUD in $\mathcal{C}$. But the edges of D are morphisms in $(A \downarrow G)$, so the f_j form a cone over GUD. Thus, we obtain a unique factorisation $f : A \to GL$ such that $(G\lambda_j)f = f_j$ for all j. In other words, we have a unique lifting of L to an object (L, f) of $(A \downarrow G)$ which makes the λ_j into a cone over D with apex (L, f). Any cone over D with apex (M, g) becomes a cone over UD with apex M by forgetting the structure map, so we get a unique $h : M \to L$, and this becomes a morphism in $(A \downarrow G)$ as both $(Gh)g$ and f are factorisations through L of the same cone over UD. □

Lemma. Let $\mathcal{C}$ be a category. Specifying an initial object of $\mathcal{C}$ is equivalent to specifying a limit for the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, considered as a diagram of shape $\mathcal{C}$ in $\mathcal{C}$.

Proof. First, suppose we have an initial object I in $\mathcal{C}$. Then the unique morphisms $I \to A$ form a cone over $1_{\mathcal{C}}$, and it is a limit, because for any other cone $(B, (\lambda_A : B \to A))$, then λ_I is the unique factorisation as required. Conversely, suppose $(I, (\lambda_A : I \to A))$ is a limit for $1_{\mathcal{C}}$. Then certainly I is *weakly initial*: it has at least one morphism to any other object, given by λ_A. For any morphism $f : I \to A$, it is an edge of the diagram, so $f\lambda_I = \lambda_A$, so it suffices to show that λ_I is the identity morphism. Using the same equation with $f = \lambda_A$, we obtain $\lambda_A\lambda_I = \lambda_A$, so λ_I is a factorisation of the limit cone through itself. As this factorisation must be unique, we must have $\lambda_I = 1_I$. $\square$

Proposition (primitive adjoint functor theorem)**.** If $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits, then G has a left adjoint.

Proof. The categories $(A \downarrow G)$ have all limits, and in particular they have initial objects, so G has a left adjoint. $\square$

4.5 General adjoint functor theorem

Theorem (general adjoint functor theorem)**.** Suppose $\mathcal{D}$ is complete and locally small. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if G preserves small limits and satisfies the *solution-set condition*: given any $A \in \operatorname{ob} \mathcal{C}$, there is a set $\{f_i : A \to GB_i\}_{i \in I}$ such that every $f : A \to GB$ factors as

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GB$$

for some $i \in I$ and $g : B_i \to B$. This set I is called a solution-set at A.

The solution-set condition can be equivalently phrased as the assertion that the categories $(A \downarrow G)$ all have *weakly initial* sets of objects: every object of $(A \downarrow G)$ admits a morphism from a member of the solution set.

Proof. If $F \dashv G$, then G preserves all limits that exist in its domain, so in particular it preserves small limits, and $\{\eta_A : A \to GFA\}$ is a solution-set at A for any A. Now suppose $A \in \operatorname{ob} \mathcal{C}$. Then $(A \downarrow G)$ is complete, and is locally small as morphisms $(B, f) \to (B', f')$ in $(A \downarrow G)$ are a subset of $\mathcal{D}(B, B')$. We must then show that if $\mathcal{A}$ is complete and locally small and has a weakly initial set of objects $\{S_i \mid i \in I\}$, then it has an initial object; then, setting $\mathcal{A} = (A \downarrow G)$ and using the solution-set as the weakly initial set, the result follows.

First, we form the product $P = \prod_{i \in I} S_i$. The set $\{P\}$ is weakly initial since we have morphisms $\pi_i : P \to S_i$ for all i. Now consider the diagram $P \rightrightarrows P$ whose edges are all endomorphisms of P. By assumption, let $i : I \to P$ be a limit for this diagram; this is an equaliser over a family of morphisms. Then I is weakly initial. For a parallel pair $f, g : I \rightrightarrows C$, we have an equaliser $e : E \to I$, and can choose some $h : P \to E$. Then we have the endomorphisms ieh and 1_P of P. Thus $iehi = 1_P i = i$, but i is monic, so $ehi = 1_I$. Hence e is a split epimorphism, and hence $f = g$. $\square$

Example. (i) Consider the forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$. Note that **Gp** is complete and locally small, and U creates small limits so in particular it preserves them. Given a set A, any function $f : A \to UG$ can be factored as

$$A \longrightarrow UG' \longrightarrow UG$$

where G' is the subgroup generated by $\{f(a) \mid a \in A\}$. Note that the cardinality of G' is at most $\max(\aleph_0, |A|)$, so we can fix a set B of this cardinality and consider all possible subsets of B, all possible group structures on those sets, and all possible functions $A \to B'$; these form a solution-set at A. Hence, free groups exist. Note that the cardinality bound on G' requires most of the technology needed to explicitly construct free groups.

(ii) Let **CLat** be the category of complete lattices. The forgetful functor $U : \mathbf{CLat} \to \mathbf{Set}$ creates all small limits; this can be seen in the same way as was shown with the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$. In 1964, A. Hales proved that there are arbitrarily large complete lattices with only three generators. Hence U has no solution set at $A = \{a, b, c\}$. Note that U is representable, or equivalently, $(1 \downarrow U)$ has an initial object. If **CLat** had all coproducts, we would be able to form initial objects for $(A \downarrow U)$, as every set is a coproduct of singletons. But **CLat** does not have even finite coproducts.

4.6 Special adjoint functor theorem

Definition. Let $A \in \operatorname{ob} \mathcal{C}$. A *subobject* of A is a monomorphism with codomain A; dually, a *quotient* of A is an epimorphism with domain A. The subobjects of A in $\mathcal{C}$ form a preorder $\operatorname{Sub}_{\mathcal{C}}(A)$ by setting $m \leq m'$ when m factors through m'. $\mathcal{C}$ is *well-powered* if $\operatorname{Sub}_{\mathcal{C}}(A)$ is equivalent to a (small) poset for any A. Dually, we say $\mathcal{C}$ is *well-copowered*.

Example. **Set** is well-powered, since every monomorphism is isomorphic to a subset inclusion; the power-set axiom encodes this fact. **Set** is also well-copowered, because quotients correspond to equivalence relations up to isomorphism, there is only a set of equivalence relations on a given object A.

Lemma. Let

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, tail, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

be a pullback square where f is monic. Then k is also monic.

Informally, monomorphisms are stable under pullback.

Proof. Let $\ell, m : D \rightrightarrows P$ be such that $k\ell = km$. Then $fhl = gk\ell = gkm = fhm$, but f is a monomorphism, so $hl = hm$.

$$\begin{tikzcd}
D \arrow[dr, shift left, "\ell"] \arrow[dr, shift right, "m"'] & & \\
& P \arrow[r, "h"] \arrow[d, "k"'] & A \arrow[d, tail, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

So ℓ and m are both factorisations of $(h\ell, k\ell)$ through the pullback, so $\ell = m$. □

Theorem. Let $\mathcal{C}, \mathcal{D}$ be locally small, and suppose that $\mathcal{D}$ is complete, well-powered, and has a coseparating set. Then a functor $G : \mathcal{D} \to \mathcal{C}$ preserves all small limits if and only if it has a left adjoint.

Proof. As above, any functor with a left adjoint preserves all limits that exist. For the other direction, fix an object A and consider the category $(A \downarrow G)$, which is complete and locally small. Note that the forgetful functor $(A \downarrow G) \to \mathcal{D}$ preserves monomorphisms, because it preserves pullbacks. Thus, one can show that $(A \downarrow G)$ is well-powered, because the subobjects of a given object (B, f) are the monomorphisms $m : B' \to B$ for which f factors through Gm. If $\{S_i\}_{i \in I}$ is a coseparating set for $\mathcal{D}$, we have a coseparating set for $(A \downarrow G)$ by taking the set of all $f : A \to GS_i$ with $i \in I$; this is a set by local smallness. This is coseparating, because given $h, k : (B, g) \rightrightarrows (B', g')$ with $h \neq k$, there is a morphism $\ell : B' \to S_i$ with $\ell h \neq \ell k$, and ℓ is a morphism $(B', g') \to (S_i, (G\ell)g')$ in $(A \downarrow G)$.

It remains to show that there is an initial object in a category $\mathcal{A}$ if it is complete, locally small, well-powered, and has a coseparating set $\{S_i\}_{i \in I}$. First, we form the product

$$P = \prod_{i \in I} S_i$$

and consider the diagram

$$\begin{tikzcd}
\bullet \arrow[dr] & \cdots & \bullet \arrow[dl] \\
& P &
\end{tikzcd}$$

whose edges are representative monomorphisms for each isomorphism class of subobjects of P. Let I be the apex of a limit cone for this 'wide pullback'. The legs of the cone are monomorphisms, using the same argument as was described for pullbacks. In particular, the composite maps $I \to P$ are monomorphisms, so I is a subobject of P. But by construction, it factors through every subobject of P, so is a minimal subobject of P.

It remains to show that I is initial. Note that if $f, g : I \rightrightarrows A$ were different monomorphisms, their equaliser $e : E \to I$ would yield a subobject of P contained in $I \to P$, so it would be an isomorphism, giving $f = g$. For an arbitrary object $A \in \operatorname{ob} \mathcal{A}$, form the product

$$Q = \prod_{(i,f)} S_i; \quad f : A \to S_i$$

and define $g : A \to Q$ by

$$\pi_{(i,f)} g = f$$

As the S_i form a coseparating family, g is a monomorphism. Thus A is a subobject of Q by g. There is a map $h : P \to Q$ defined by

$$\pi_{(i,f)}h = \pi_i$$

Thus we can form the pullback

$$\begin{tikzcd} B \arrow[r] \arrow[d, tail, "k"'] & A \arrow[d, tail, "g"] \\ P \arrow[r, tail, "h"'] & Q \end{tikzcd}$$

where k is a monomorphism as it is the pullback of a monomorphism. Hence B is a subobject of P, and thus factors through I.

$$\begin{tikzcd} I \arrow[r, dashed] \arrow[dr] & B \arrow[d, "k"] \\ & P \end{tikzcd}$$

Hence, we have a morphism $I \to A$ by composition. $\square$

Example. Let I : **KHaus** $\to$ **Top** be the inclusion functor. **KHaus** is closed under small products in **Top** by Tychonoff's theorem, and is closed under equalisers since the equaliser of $f, g : X \rightrightarrows Y$ is a closed subspace of X, and thus is compact and Hausdorff. Hence **KHaus** is complete, and the inclusion preserves small limits. It is clearly locally small and well-powered, since the subobjects of X are isomorphic to closed subspaces. It has a single coseparator, namely $[0, 1]$, by Urysohn's lemma. Hence, by the special adjoint functor theorem, I has a left adjoint β, which is the Stone–Čech compactification functor.

Remark. Čech's construction of β is almost identical to the construction of left adjoints given above. Given a space X, one can form

$$P = \prod_{f:X\to[0,1]} [0, 1]; \quad g : X \to P; \quad \pi_f g = f$$

which is the product of the members of coseparating set for $(X \downarrow I)$. Then, βX can be defined to be the closure of the image of g, that is, the smallest subobject of (P, g) in $(X \downarrow I)$.

The general adjoint functor theorem can also be used to construct β. To obtain a solution-set at a space X, observe that any morphism from X to a compact Hausdorff space IY factors as $X \to IY' \to IY$ where Y' is the closure of $X' = \{f(x) \mid x \in X\}$. One can show that if Y' is Hausdorff and X' is dense in Y', then $|Y'| \leq 2^{2^{|X'|}}$.

5 Monads

5.1 Definition

Suppose $F \dashv G$ is an adjunction with $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ is a well-understood category, but $\mathcal{D}$ is not. We can study $\mathcal{D}$ indirectly inside the context of $\mathcal{C}$ by using the adjunction. We have the composite $T = GF : \mathcal{C} \to \mathcal{C}$, and we have the unit $\eta : 1_{\mathcal{C}} \to T$. The counit is not directly accessible from $\mathcal{C}$, but we have $\mu = G\epsilon_F : T^2 \to T$. The triangular identities give rise to identities linking η and μ.

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd}$$

In addition, naturality of ϵ gives

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Definition. A *monad* on a category $\mathcal{C}$ is a triple $\mathbb{T} = (T, \eta, \mu)$ where T is a functor $\mathcal{C} \to \mathcal{C}$, and $\eta : 1_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are natural transformations satisfying the following commutative diagrams.

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

η is the *unit* of the monad, and μ is the *multiplication* of the monad.

The dual notion is called a *comonad*.

Example. (i) Let M be a monoid. The functor $M \times (-) : \mathbf{Set} \to \mathbf{Set}$ has a monad structure. The unit $\eta_A : A \to M \times A$ maps each a to $(1, a)$, and the multiplication $\mu_A : M \times M \times A \to M \times A$ maps (m, m', a) to (mm', a). These maps are natural. The required commutative diagrams encode precisely the left and right unit laws and the associativity law of a monoid. In fact, monoids correspond precisely to monads on **Set** whose underlying functors have right adjoints.

(ii) Let $P : \mathbf{Set} \to \mathbf{Set}$ be the covariant power-set functor. This can be given a monad structure. The unit $\eta_A : A \to PA$ maps a to its singleton $\{a\}$, and the multiplication $\mu_A : PPA \to PA$ is the union operation mapping S to $\bigcup S$. One can check that the required laws are satisfied.

These examples both arise as a result of adjunctions. Example (a) arises from the free M-set functor $F : \mathbf{Set} \to [M, \mathbf{Set}]$ and the forgetful functor $U : [M, \mathbf{Set}] \to \mathbf{Set}$, where $F \dashv U$. For example (b), there is a forgetful functor $U : \mathbf{CSLat} \to \mathbf{Set}$ from the category of complete (join-)semilattices. This has a left adjoint $P : \mathbf{Set} \to \mathbf{CSLat}$, which is the free complete semilattice on A. Indeed, given any $f : A \to UB$, there is a unique extension of f to a join-preserving map $\overline{f} : PA \to B$ given by

$$\overline{f}(A') = \bigvee \{f(a') \mid a' \in A'\}$$

Note that an M-set is a set A equipped with a map $\alpha : M \times A \to A$, and a complete semilattice is a set A equipped with a map $\bigvee : PA \to A$. So the elements of the other category can be defined in terms of the monad.

This holds in general: every monad arises from an adjunction. We present two constructions.

5.2 Eilenberg–Moore algebras

Definition. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. An *Eilenberg–Moore algebra* or $\mathbb{T}$*-algebra* is a pair (A, α) where A is an object in $\mathcal{C}$, and $\alpha : TA \to A$ is a morphism satisfying

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

A homomorphism of algebras $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

This forms a category of $\mathbb{T}$-algebras, denoted $\mathcal{C}^{\mathbb{T}}$.

Proposition. The forgetful functor $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ has a left adjoint $F^{\mathbb{T}}$, and the adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ induces the monad $\mathbb{T}$ on $\mathcal{C}$.

Proof. We define the *free algebra* of an object A to be $F^{\mathbb{T}}A = (TA, \mu_A)$. This defines an algebra structure on TA for every A by the monad laws. For $f : A \to B$, we define $F^{\mathbb{T}}f = Tf$; this is a homomorphism by naturality of μ. This is functorial as T is functorial.

We have $G^{\mathbb{T}}F^{\mathbb{T}} = T$. For the unit of the adjunction, we use the unit of the monad η. For the counit, we define

$$\mu_{(A,\alpha)} = \alpha : F^{\mathbb{T}}A \to (A, \alpha)$$

This is a homomorphism by the definition of an algebra, and it is a natural transformation by the definition of homomorphisms of algebras. It suffices to verify the triangular identities, which follows from the remaining unused diagrams. One can check that the multiplication induced by this monad is equal to that of $\mathbb{T}$. □

5.3 Kleisli categories

If $F \dashv G$ with $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is an adjunction inducing $\mathbb{T}$, then $F' \dashv G'$ with $F' : \mathcal{C} \to \mathcal{D}'$ and $G' : \mathcal{D}' \to \mathcal{C}$, where $\mathcal{D}'$ is the full subcategory of $\mathcal{D}$ on objects in the image of F. Thus, when finding a construction for $\mathcal{D}$, we can assume that F is surjective (or, indeed, bijective) on objects. Then, the morphisms $FA \to FB$ must correspond to morphisms $A \to GFB$ under the adjunction, but $GF = T$.

Definition. Let $\mathbb{T} = (T, \mu, \eta)$ be a monad on $\mathcal{C}$. The *Kleisli category* $\mathcal{C}_{\mathbb{T}}$ is the category where the objects are precisely the objects of $\mathcal{C}$, and the morphisms from A to B in $\mathcal{C}_{\mathbb{T}}$ are the morphisms $A \to TB$ in $\mathcal{C}$. To avoid confusion, we will denote morphisms from A to B in this category by $A \dashrightarrow B$. The identity $A \dashrightarrow A$ is $\eta_A : A \to TA$. The composite of

$$\begin{tikzcd} A \arrow[r, dotted, "f"] & B \arrow[r, dotted, "g"] & C \end{tikzcd}$$

is

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$$

These satisfy the unit and associativity laws.

$$\begin{tikzcd} A \arrow[r, "f"] & TB \arrow[r, "T\eta_B"] \arrow[dr, "1_{TB}"'] & T^2B \arrow[d, "\mu_B"] \\ & & TB \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & TA \arrow[d, "Tf"] \\ TB \arrow[r, "\eta_{TB}"] \arrow[dr, "1_{TB}"'] & T^2B \arrow[d, "\mu_B"] \\ & TB \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] & TB \arrow[r, "Tg"] & T^2C \arrow[r, "T^2h"] \arrow[d, "\mu_C"] & T^3D \arrow[r, "T\mu_D"] \arrow[d, "\mu_{TD}"] & T^2D \arrow[d, "\mu_D"] \\ & & TC \arrow[r, "Th"'] & T^2D \arrow[r, "\mu_D"'] & TD \end{tikzcd}$$

where in the last diagram, the upper composite is $(hg)f$ and the lower composite is $h(gf)$ in $\mathcal{C}_{\mathbb{T}}$.

Proposition. There is an adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ where $F_{\mathbb{T}} : \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ and $G_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ that induces the monad $\mathbb{T}$.

Proof. We define $F_{\mathbb{T}}A = A$, and for $f : A \to B$, define $F_{\mathbb{T}}f = \eta_B f$. This preserves identities as $1_{F_{\mathbb{T}}A} = \eta_A$, and preserves composites since

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "\eta_B"] \arrow[d, "g"] & TB \arrow[d, "Tg"'] & T^2C \arrow[d, "\mu_C"] \\ & C \arrow[r, "\eta_C"] & TC \arrow[r, "1_{TC}"'] \arrow[ur, "T\eta_C"] & TC \end{tikzcd}$$

commutes. For $G_{\mathbb{T}}$, we define $G_{\mathbb{T}}A = TA$, and for $f : A \dashrightarrow B$, we define $G_{\mathbb{T}}f$ to be the composite

$$TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB$$

Note that $G_{\mathbb{T}}$ preserves identities by the unit law and preserves composites as

$$\begin{tikzcd} TA \arrow[r, "Tf"] & T^2B \arrow[r, "T^2g"] \arrow[d, "\mu_B"] & T^3C \arrow[r, "T\mu_C"] \arrow[d, "\mu_{TC}"] & T^2C \arrow[d, "\mu_C"] \\ & TB \arrow[r, "Tg"'] & T^2C \arrow[r, "\mu_C"'] & TC \end{tikzcd}$$

commutes. Then $G_{\mathbb{T}}$ is a functor, and $G_{\mathbb{T}}F_{\mathbb{T}} = T$. The unit of the adjunction is the unit of the monad η. For the counit $\epsilon_A : TA = F_{\mathbb{T}}G_{\mathbb{T}}A \rightsquigarrow A$, we use the identity 1_{TA}. This is natural, as given $f : A \rightsquigarrow B$, the diagram

$$\begin{tikzcd} TA \arrow[r, dotted, "F_{\mathbb{T}}G_{\mathbb{T}}f"] \arrow[d, dotted, "\epsilon_A"'] & TB \arrow[d, dotted, "\epsilon_B"] \\ A \arrow[r, dotted, "f"'] & B \end{tikzcd}$$

commutes, as the paths are

$$TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB \xrightarrow{\eta_{TB}} T^2B \xrightarrow{\mu_B} TB$$

and

$$TA \xrightarrow{Tf} T^2B \xrightarrow{\mu_B} TB$$

which coincide. One can show that both triangular identities reduce to a unit law. It suffices to verify that the multiplication of the induced monad is correct. The multiplication law is $G_{\mathbb{T}}\epsilon_{F_{\mathbb{T}}A}$, which is

$$T^2A \xrightarrow{T1_{TA}} T^2A \xrightarrow{\mu_A} TA$$

which is equal to μ_A, as required. □

5.4 Comparison functors

Definition. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. Then $\mathrm{Adj}(\mathbb{T})$ is the category of adjunctions $F \dashv G$ which induce $\mathbb{T}$, where the morphisms $F \dashv G$ to $F' \dashv G'$ are the functors $K : \mathcal{D} \to \mathcal{D}'$ satisfying $KF = F'$ and $G'K = G$.

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "F"'] \arrow[dr, "F'"] & \\ \mathcal{D} \arrow[rr, "K"] \arrow[dr, "G"'] & & \mathcal{D}' \arrow[dl, "G'"] \\ & \mathcal{C} & \end{tikzcd}$$

Theorem. The Kleisli adjunction $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ is initial in $\mathrm{Adj}(\mathbb{T})$, and the Eilenberg–Moore adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal in $\mathrm{Adj}(\mathbb{T})$.

Proof. We will first do the case of the Eilenberg–Moore adjunction. Let $F \dashv G$ be an adjunction inducing $\mathbb{T}$. We define $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ by $KB = (GB, G\epsilon_B)$. This is an algebra by the triangular identities and naturality of ϵ. On morphisms $f : B \to C$ in $\mathcal{D}$, we define $Kg = Gg$, which is a homomorphism as ϵ is a natural transformation. Clearly $G^{\mathbb{T}}K = G$, and $KFA = (GFA, G\epsilon_{FA}) = F^{\mathbb{T}}A$, and for $f : A \to A'$, $KFf = GFf = Tf = F^{\mathbb{T}}f$. So K is a morphism of $\mathrm{Adj}(\mathbb{T})$.

For uniqueness, suppose K' were another such morphism. Then $K'B = (GB, \beta_B)$, and $K'g = Gg$ for $g : B \to C$. Note that β must be a natural transformation $GFG \to G$. Also, $\beta_{FA} = G\epsilon_{FA}$ for all A, as

$K'F = F^{\mathbb{T}}$. But we have naturality squares

$$\begin{tikzcd} GFGFGB \arrow[r, "GFG\epsilon_B"] \arrow[d, shift right, "\beta_{FGB}"'] \arrow[d, shift left, "G\epsilon_{FGB}"] & GFGB \arrow[d, shift right, "\beta_B"'] \arrow[d, shift left, "G\epsilon_B"] \\ GFGB \arrow[r, "G\epsilon_B"'] & GB \end{tikzcd}$$

where the left edges are equal and the top edge is a split epimorphism, so the right edges are equal. Thus K is unique.

Given an adjunction $F \dashv G$ inducing $\mathbb{T}$, we define $H : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ by $HA = FA$, and for $f : A \rightsquigarrow B$, define Hf to be the composite

$$FA \xrightarrow{Ff} FGFB \xrightarrow{\epsilon_{FB}} FB$$

This is functorial. Indeed, for $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$, $H(gf)$ is the upper composite and $(Hg)(Hf)$ is the lower composite in the following diagram.

$$\begin{tikzcd} FA \arrow[r, "Ff"] & FGFB \arrow[r, "FGFg"] \arrow[d, "\epsilon_{FB}"] & FGFGFC \arrow[r, "FG\epsilon_{FC}"] \arrow[d, "\epsilon_{FGFC}"] & FGFC \arrow[d, "\epsilon_{FC}"] \\ & FB \arrow[r, "Fg"'] & FGFC \arrow[r, "\epsilon_{FC}"'] & FC \end{tikzcd}$$

Then $HF_{\mathbb{T}}(f) = \epsilon_{FB}(F\eta_B)(Ff) = Ff$. Moreover, $GHA = GFA = TA = G_{\mathbb{T}}A$, and for $f : A \rightsquigarrow B$, GFf is the composite

$$GFA \xrightarrow{GFf} GFGFB \xrightarrow{\mu_B} GFB$$

which is the definition of $G_{\mathbb{T}}(f)$. Thus H is a morphism of $\mathrm{Adj}(\mathbb{T})$. If $H' : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ were another such morphism, then since $H'F_{\mathbb{T}} = F$, we must have $H'A = FA$ for all A. Note that for $f : A \rightsquigarrow B$, Hf is the transpose of $f : A \to GFB$ across $F \dashv G$. Since H' commutes with G and $G_{\mathbb{T}}$, and $F \dashv G$ and $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ have the same unit η, H' must send the transpose $f : A \rightsquigarrow B$ of $f : A \to GFB$ to its transpose across $F \dashv G$, which is precisely the action of H on morphisms. Hence $H' = H$. □

Definition. The functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is called the *Eilenberg–Moore comparison functor*. Similarly, the functor $H : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ is called the *Kleisli comparison functor*.

Remark. Note that $\mathcal{C}_{\mathbb{T}}$ has coproducts if $\mathcal{C}$ does, since $F_{\mathbb{T}}$ preserves them and is bijective on objects. However, it has few other limits or colimits in general. In contrast, $\mathcal{C}^{\mathbb{T}}$ inherits many limits and colimits from $\mathcal{C}$.

Proposition.
(i) The forgetful functor $G = G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates any limits which exist in $\mathcal{C}$.
(ii) If $\mathcal{C}$ has colimits of shape J, then $G = G^{\mathbb{T}}$ creates colimits of shape J if and only if T preserves them.

Proof. Part (i). Let $D : J \to \mathcal{C}^{\mathbb{T}}$ be a diagram of shape J. Write $D(j) = (GD(j), \delta_j)$ for $j \in \mathrm{ob}\, J$. Let $(L, (\lambda_j : L \to GD(j))_{j \in \mathrm{ob}\, J})$ be a limit for GD in $\mathcal{C}$. Then $(TL, (T\lambda_j)_{j\in \mathrm{ob}\, J})$ is a cone over TGD, so

$(TL, (\delta(T\lambda_j))_{j \in \mathrm{ob}\, J})$ is a cone over TGD, and induces a unique $\theta : TL \to L$ making squares of the form

$$\begin{tikzcd} TL \arrow[r, "T\lambda_j"] \arrow[d, "\theta"'] & TGD(j) \arrow[d, "\delta_j"] \\ L \arrow[r, "\lambda_j"'] & GD(j) \end{tikzcd}$$

commute for each j. Note that θ is an algebra structure on L, since the required diagrams commute by uniqueness of factorisation through limits. It is the unique algebra structure on L which make the λ_j into a cone in $\mathcal{C}^{\mathbb{T}}$, and one can easily show it is a limit cone.

Part (ii). In the forward direction, if G creates colimits of shape J, then it certainly preserves them, as they exist in both categories. But F preserves all colimits, so $T = GF$ preserves them. Given $D : J \to \mathcal{C}^{\mathbb{T}}$ and a colimit cone $\lambda_j : GD(j) \to L$ under GD, we know that $T\lambda_j : TGD(j) \to TL$ is a colimit cone, so there is a unique $\theta : TL \to L$ satisfying $\theta(T\lambda_j) = \lambda_j \delta_j$ for all j, and θ is an algebra structure since TTL is also a colimit. Hence (L, θ) is a colimit for D in $\mathcal{C}^{\mathbb{T}}$. $\square$

Remark. One can show that $\mathcal{C}^{\mathbb{T}}$ has colimits of any shape which exist in $\mathcal{C}$, provided that it has *reflexive coequalisers*.

5.5 Monadic adjunctions

It can be useful to know, for an arbitrary adjunction, if the Eilenberg–Moore comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is part of an equivalence of categories. Note that the Kleisli comparison functor H is always full and faithful, so is part of an equivalence if and only if it is essentially surjective, and since its action on objects is F, this holds if and only if F is essentially surjective.

Definition. An adjunction $F \dashv G$ is *monadic*, or the right adjoint G is *monadic*, if K is part of an equivalence.

Lemma. Let $F \dashv G$ be an adjunction inducing the monad $\mathbb{T}$, and suppose that for every $\mathbb{T}$-algebra (A, α), the pair

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] & FA \end{tikzcd}$$

has a coequaliser in $\mathcal{D}$. Then the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L.

Proof. Let $\lambda_{(A,\alpha)} : FA \to L(A, \alpha)$ be a coequaliser for $F\alpha, \epsilon_{FA}$. We can make L into a functor $\mathcal{C}^{\mathbb{T}} \to \mathcal{D}$. Given $f : (A, \alpha) \to (B, \beta)$, the composite $\lambda_{(B,\beta)}(Ff)$ coequalises $F\alpha$ and ϵ_{FA}, so it induces a unique map $Lf : L(A, \alpha) \to L(B, \beta)$. This makes L into a functor by uniqueness.

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] \arrow[d, "FGFf"'] & FA \arrow[r, "\lambda_{(A,\alpha)}"] \arrow[d, "Ff"] & L(A, \alpha) \arrow[d, "Lf"] \\ FGFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\epsilon_{FB}"'] & FB \arrow[r, "\lambda_{(B,\beta)}"'] & L(B, \beta) \end{tikzcd}$$

For any object B of $\mathcal{D}$, morphisms $L(A, \alpha) \to B$ correspond to morphisms $f : FA \to B$ satisfying $f(F\alpha) = f\epsilon_{FA}$. If $\overline{f} : A \to GB$ is the transpose of f across $F \dashv G$, then by naturality, the transpose of $f(F\alpha)$ is $\overline{f}\alpha$, and the transpose of $f\epsilon_{FA}$ is Gf since ϵ_{FA} transposes to 1_{GFA}. But we have $f = \epsilon_B(F\overline{f})$, so $(G\epsilon_B)(GF\overline{f}) = (G\epsilon_B)(T\overline{f})$. Thus $f(F\alpha) = f(\epsilon_{FA})$ if and only if $\overline{f}\alpha = (G\epsilon_B)(T\overline{f})$, which is to say that $\overline{f}$ is an algebra homomorphism $(A, \alpha) \to (GB, G\epsilon_B) = KB$. Naturality of this bijection follows from the fact that the map $f \mapsto \overline{f}$ is natural, so $L \dashv K$ as required. □

Definition. A parallel pair $f, g : A \rightrightarrows B$ is *reflexive* if there exists $r : B \to A$ such that $fr = gr = 1_B$.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
B \arrow[u, "r"] \arrow[ur, bend right, "1_B"'] &
\end{tikzcd}$$

Note that the parallel pair

$$\begin{tikzcd}
FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] & FA
\end{tikzcd}$$

is a reflexive pair, and the common right inverse is $r = F\eta_A$.

Definition. A *split coequaliser diagram* is a diagram

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left, "t"] \arrow[r, shift left, "h"] & C \arrow[l, bend left, "s"]
\end{tikzcd}$$

such that $hf = hg, hs = 1_C, gt = 1_B, ft = sh$. That is, h has equal composites with f and g, and the following diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "g"] & B \arrow[r, "h"] & C \\
B \arrow[u, "t"] \arrow[ur, "1_B"'] & C \arrow[u, "s"] \arrow[ur, "1_C"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "t"] \arrow[d, "h"'] & A \arrow[d, "f"] \\
C \arrow[r, "s"'] & B
\end{tikzcd}$$

The equations $hs = 1_C, gt = 1_B$ enforce that s is a section of h, and t is a section of g. The equation $ft = sh$ enforces that the two non-identity paths from B to itself coincide.

Note that this implies that h is a coequaliser of f and g. Indeed, if $k : B \to D$ satisfies $kf = kg$, then $k = kgt = kft = ksh$, so k factors through h. Moreover, this factorisation is unique as h is split epic. Any functor preserves split coequaliser diagrams.

Definition. Given a functor $G : \mathcal{D} \to \mathcal{C}$, we say that a parallel pair $f, g : A \rightrightarrows B$ in $\mathcal{D}$

is *G-split* if there is a split coequaliser diagram

$$\begin{tikzcd} GA \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GB \arrow[l, bend left, "t"] \arrow[r, shift left, "h"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

in $\mathcal{C}$.

Note that the pair

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] & FA \end{tikzcd}$$

is G-split, as

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "GF\alpha"] \arrow[r, shift right, "G\epsilon_{FA}=\mu_A"'] & GFA \arrow[l, bend left, "\eta_{GFA}"] \arrow[r, shift left, "\alpha"] & C \arrow[l, bend left, "\eta_A"] \end{tikzcd}$$

is a split coequaliser diagram.

Theorem (Beck's precise monadicity theorem)**.** A functor $G : \mathcal{D} \to \mathcal{C}$ is monadic if and only if G has a left adjoint and creates coequalisers of G-split pairs.

Theorem (Beck's crude monadicity theorem)**.** Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, and G reflects isomorphisms. Suppose further that $\mathcal{D}$ has and G preserves reflexive coequalisers. Then G is monadic.

We prove both theorems together.

Proof. First, suppose $G : \mathcal{D} \to \mathcal{C}$ is monadic. Then G has a left adjoint by definition. It suffices to show that $G^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates coequalisers of $G^{\mathbb{T}}$-split pairs. This follows from the argument of a previous lemma: if $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ are algebra homomorphisms, and

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend left, "t"] \arrow[r, shift left, "h"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

is a split coequaliser, then since the coequaliser is preserved by T and T^2, C acquires a unique algebra structure $\gamma : TC \to C$ such that h is a coequaliser in $\mathcal{C}^{\mathbb{T}}$.

For the converse, either set of assumptions ensures that $\mathcal{D}$ has coequalisers of parallel pairs of the form

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\epsilon_{FA}"'] & FA \end{tikzcd}$$

so the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ has a left adjoint L. We must now show that the unit and counit of $L \dashv K$ are isomorphisms. The unit $(A, \alpha) \to KL(A, \alpha)$ is the unique factorisation of

$G\lambda_{(A,\alpha)} : GFA \to GL(A,\alpha)$ through the ($G^{\mathbb{T}}$-split) coequaliser $\alpha : GFA \to A$ of $GF\alpha, G\epsilon_{FA} : GFGFA \rightrightarrows GFA$ in $\mathcal{C}^{\mathbb{T}}$. But either set of hypotheses implies that G preserves the coequaliser of $F\alpha, \epsilon_{FA}$, so the factorisation is an isomorphism. The counit $LKB \to B$ is the unique factorisation of $\epsilon_B : FGB \to B$ through $\lambda_{KB} : FGB \to LKB$. The hypothesis in the precise theorem implies directly that ϵ_B is a coequaliser of $FG\epsilon_B, \epsilon_{GFB}$, because the pair is G-split. From the hypotheses of the crude theorem, we can see that both ϵ_B and λ_{KB} map to coequalisers in $\mathcal{C}$, so the counit maps to an isomorphism in $\mathcal{C}$, so it is an isomorphism as G reflects isomorphisms. □

Remark. (i) Let J be the finite category

$$\begin{tikzcd} A \arrow[loop above, "s"] \arrow[loop below, "t"] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & B \arrow[l, "r"] \end{tikzcd}$$

with $fr = gr = 1_B, rf = s, rg = t$, then a diagram D of this shape is a reflexive pair. A cone under it is determined by $h : DB \to L$, which must satisfy $h(Df) = h(Dg)$. A colimit for this diagram is a coequaliser for f, g.

(ii) All small (respectively finite) colimits can be constructed from small (respectively finite) coproducts and reflexive coequalisers. The pair $f, g : P \rightrightarrows Q$ in the proof form a coreflexive pair, with common left inverse $r : Q \to P$ given by $\pi_j r = \pi_{1_j}$ for all j.

(iii) Given a reflexive pair $f, g : A \rightrightarrows B$, a morphism $h : B \to C$ is a coequaliser for it if and only if the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ B \arrow[r, "h"'] & C \end{tikzcd}$$

is a pushout, since any cone under the span given by f and g has its two legs equal. The dual of this statement has already been proven.

(iv) In any cartesian closed category, reflexive coequalisers commute with finite products: if the following are reflexive coequaliser diagrams,

$$\begin{tikzcd} A_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "g_1"'] & B_1 \arrow[r, "h_1"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[r, shift left, "f_2"] \arrow[r, shift right, "g_2"'] & B_2 \arrow[r, "h_2"] & C_2 \end{tikzcd}$$

then the following diagram is also a coequaliser.

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, shift left, "f_1 \times f_2"] \arrow[r, shift right, "g_1 \times g_2"'] & B_1 \times B_2 \arrow[r, "h_1 \times h_2"] & C_1 \times C_2 \end{tikzcd}$$

Indeed, consider the diagram

$$\begin{tikzcd}
A_1 \times A_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B_1 \times A_2 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & C_1 \times A_2 \arrow[d, shift left] \arrow[d, shift right] \\
A_1 \times B_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & B_1 \times B_2 \arrow[r] \arrow[d] & C_1 \times B_2 \arrow[d] \\
A_1 \times C_2 \arrow[r, shift left] \arrow[r, shift right] & B_1 \times C_2 \arrow[r] & C_1 \times C_2
\end{tikzcd}$$

All rows and columns are coequalisers, since functors of the form $(-)\times D$ preserve coequalisers. It then follows that the lower right square is a pushout. By reflexivity, if $k : B_1 \times B_2 \to D$ coequalises

$$f_1 \times f_2, g_1 \times g_2 : A_1 \times A_2 \rightrightarrows B_1 \times B_2$$

then it also coequalises $B_1 \times A_2 \rightrightarrows B_1 \times B_2$ and $A_1 \times B_2 \rightrightarrows B_1 \times B_2$, as they both factor through the diagonal pair. Therefore, it factors through the top and left edges of the lower right square, and hence through its diagonal.

Example. (i) The forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$ satisfies the hypotheses of the crude monadicity theorem. Indeed, it has a left adjoint and reflects isomorphisms, and it creates reflexive coequalisers. Given a reflexive pair $f, g : A \rightrightarrows B$ in $\mathbf{Gp}$, consider its coequaliser $h : UB \to C$ in $\mathbf{Set}$. As reflexive coequalisers commute with products in $\mathbf{Set}$,

$$\begin{tikzcd}
UA \times UA \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & UB \times UB \arrow[r] & C \times C
\end{tikzcd}$$

is a coequaliser. So we obtain a binary operation $C \times C \to C$ making h into a homomorphism, C into a group, and h a coequaliser in $\mathbf{Gp}$. The same procedure applies for many other algebraic structures, such as rings, modules over a given ring, and lattices. For infinitary algebraic categories such as complete semilattices and complete lattices, we can use the precise monadicity theorem whenever a left adjoint exists.

(ii) Any reflection is monadic. If $I : \mathcal{D} \to \mathcal{C}$ is the inclusion of a reflective subcategory and $f, g : A \rightrightarrows B$ is an I-split pair in $\mathcal{D}$, then the splitting $t : B \to A$ belongs to $\mathcal{D}$, and so its composite $ft = sh$ also lies in $\mathcal{D}$. But $\mathcal{D}$ is closed under limits that exist in $\mathcal{C}$, so in particular it is closed under splittings of idempotents.

(iii) Consider the composite adjunction

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift left, "F"] & \mathbf{AbGp} \arrow[l, shift left, "U"] \arrow[r, shift left, "L"] & \mathbf{tfAbGp} \arrow[l, shift left, "I"]
\end{tikzcd}$$

Both factors are monadic: we have already shown that $F \dashv U$ is monadic, and $L \dashv I$ is a reflection. However, the composite $LF \dashv UI$ is not monadic. Indeed, free abelian groups are torsion-free, so the monad induced by the composite adjunction coincides with that induced by $F \dashv U$.

(iv) The contravariant power-set functor $P^\star : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is monadic as it satisfies the hypotheses of the crude monadicity theorem. Its left adjoint is $P^\star : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$, and it reflects isomorphisms. Let

$$\begin{tikzcd}
A \arrow[r, "e"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}$$

be a coreflexive equaliser in **Set**. Then the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "g"] \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

is a pullback. Thus, the composite

$$PB \xrightarrow{P^{\star}e} PA \xrightarrow{Pe} PB$$

coincides with

$$PB \xrightarrow{Pg} PC \xrightarrow{P^{\star}f} PB$$

Also, $(P^{\star}e)(Pe) = 1_{PA}$ and $(P^{\star}g)(Pg) = 1_{PB}$, so we obtain the following split coequaliser diagram in **Set**.

$$\begin{tikzcd} PC \arrow[r, shift left, "P^{\star}f"] \arrow[r, shift right, "P^{\star}g"'] & PB \arrow[l, bend left=60, "Pg"] \arrow[r, "P^{\star}e"] & PA \arrow[l, bend left, "Pe"] \end{tikzcd}$$

(v) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is not monadic. The monad induced by $D \dashv U$ is $1_{\mathbf{Set}}$, and the unit and multiplication are the identity natural transformations. Hence its category of algebras is isomorphic to **Set**. This example demonstrates that reflection of isomorphisms is necessary for the crude theorem.

(vi) The composite

$$\begin{tikzcd} \mathbf{Set} \arrow[r, shift left, "D"] & \mathbf{Top} \arrow[l, shift left, "U"] \arrow[r, shift left, "\beta"] & \mathbf{KHaus} \arrow[l, shift left, "I"] \end{tikzcd}$$

is monadic, where β is the Stone–Čech compactification functor; we will prove this using the precise monadicity theorem. Consider a UI-split pair $f, g : X \rightrightarrows Y$ in **KHaus**.

$$\begin{tikzcd} UX \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UY \arrow[l, bend left=60, "t"] \arrow[r, "h"] & Z \arrow[l, bend left, "s"] \end{tikzcd}$$

There is a unique topology on Z making h into a coequaliser in **Top**, which is the quotient topology. This is compact as it is a continuous image of the compact space Y. Hence h will be a coequaliser in **KHaus** if and only if this topology is Hausdorff. Note that the quotient topology is the only possible candidate topology on Z that could make h into a morphism in **KHaus**.

It is a general fact that for every compact Hausdorff space Y and equivalence relation $S \subseteq Y \times Y$, the quotient is Hausdorff if and only if S is closed as a subset of $Y \times Y$. Suppose $(y_1, y_2) \in S$, so $h(y_1) = h(y_2)$. Then the elements $x_1 = t(y_1)$ and $x_2 = t(y_2)$ satisfy

$$g(x_1) = y_1; \quad g(x_2) = y_2; \quad f(x_1) = f(x_2)$$

and if x_1, x_2 satisfy these three equations, then $h(y_1) = h(y_2)$. Thus S is the image under $g \times g : X \times X \to Y \times Y$ of the equivalence relation R on X given by $\{(x_1, x_2) \mid f(x_1) = f(x_2)\}$. But R is closed in $X \times X$, as it is the equaliser of $f\pi_1, f\pi_2 : X \times X \rightrightarrows Y$ into a Hausdorff space, so it is compact. Hence S is compact, and thus closed.

Definition. Let $F \dashv G$ be an adjunction with $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$. Suppose that $\mathcal{D}$ has reflexive coequalisers. The *monadic tower* of $F \dashv G$ is the diagram

$$
\begin{tikzcd}
 & \vdots \\
 & (\mathcal{C}^{\mathbb{T}})^{\mathbb{S}} \arrow[dl, bend right] \arrow[d, shift left] \\
\mathcal{D} \arrow[ur, bend left] \arrow[r, shift right, "K"'] \arrow[dr, bend right, "G"] & \mathcal{C}^{\mathbb{T}} \arrow[l, shift right, "L"'] \arrow[u, shift left] \arrow[d, shift left] \\
 & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[u, shift left]
\end{tikzcd}
$$

where $\mathbb{T}$ is the monad induced by $F \dashv G$, K is the comparison functor, L is the left adjoint to K which exists as $\mathcal{D}$ has reflexive coequalisers, $\mathbb{S}$ is the monad induced by $L \dashv K$, and so on. We say that $F \dashv G$ has *monadic length n*, or that $\mathcal{D}$ has *monadic height n* over $\mathcal{C}$, if the tower reaches an equivalence after n steps.

If $F \dashv G$ is an equivalence, it has monadic length zero. Monadic length one means that $F \dashv G$ is monadic but not an equivalence, and example (iii) above has monadic length two.

6 Monoidal and enriched categories

6.1 Monoidal categories

There are many examples of categories $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $I \in \operatorname{ob} \mathcal{C}$ that turn $\mathcal{C}$ into a monoid up to isomorphism. Such a structure on a category is called a *monoidal structure*, which will be defined precisely at the end of this subsection.

Example. (i) Let $\mathcal{C}$ be a category with finite products. Let $\otimes$ be the categorical product $\times$, and let $I = 1$ be the terminal object. This is known as the *cartesian monoidal structure*. Dually, if $\mathcal{C}$ is a category with finite coproducts, it has a *cocartesian monoidal structure*, given by $\otimes = +$ and $I = 0$.

(ii) In **Met**, the different metrics on $X \times Y$ yield different monoidal structures on **Met**. Each of these have the one-point space, which is the terminal object, as the unit of the monoid.

(iii) In **AbGp**, the tensor product gives a monoidal structure, where $\mathbb{Z}$ is the unit. Recall that if A, B, C are abelian groups, then morphisms $A \otimes B \to C$ (that is, $\mathbb{Z}$-linear maps) correspond to $\mathbb{Z}$-bilinear maps $A \times B \to C$. Similarly, if R is a commutative ring, the tensor product $\otimes_R$ gives a monoidal structure on $\mathbf{Mod}_R$ with unit R. The R-linear maps $A \otimes B \to C$ correspond to R-bilinear maps $A \times B \to C$.

(iv) For any category $\mathcal{C}$, its category of endofunctors $[\mathcal{C}, \mathcal{C}]$ has a monoidal structure given by composition. The unit is the identity endofunctor $1_{\mathcal{C}}$.

(v) For posets with top and bottom elements 1 and 0, we can define the *ordinal sum* $A * B$ to be the poset obtained from their disjoint union, by identifying the top element of A with the bottom element of B. This is a monoidal structure, where the unit is the one-element poset.

Definition. A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a distinguished object I, together with three natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C); \quad \lambda_A : I \otimes A \to A; \quad \rho_A : A \otimes I \to A$$

such that the diagrams

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[rr, "\alpha_{ABC} \otimes 1_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A, B \otimes C, D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[dl, "1_A \otimes \alpha_{BCD}"] \\
& A \otimes (B \otimes (C \otimes D)) &
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commute, and $\lambda_I = \rho_I : I \otimes I \to I$. A monoidal category is *strict* if α, λ, ρ are identities.

α is called the *associator*, and λ and ρ are the *left* and *right unitors*.

These diagrams suffice to prove the commutativity of the following two diagrams.

$$\begin{tikzcd}
(I \otimes A) \otimes B \arrow[r, "\alpha_{I,A,B}"] \arrow[d, "\lambda_A \otimes 1_B"'] & I \otimes (A \otimes B) \arrow[dl, "\lambda_{A \otimes B}"] \\
A \otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes I \arrow[r, "\alpha_{A,B,I}"] \arrow[dr, "\rho_{A \otimes B}"'] & A \otimes (B \otimes I) \arrow[d, "1_A \otimes \rho_B"] \\
& A \otimes B
\end{tikzcd}$$

Note that in the category of abelian groups with the usual tensor product, the obvious choice for $\alpha_{A,B,C}$ is the map sending $(a \otimes b) \otimes c$ to $a \otimes (b \otimes c)$. However, there is also a natural isomorphism sending $(a \otimes b) \otimes c$ to $-a \otimes (b \otimes c)$. But this choice does not satisfy the pentagon equation, as a pentagon has an odd number of sides.

6.2 The coherence theorem

Given a monoidal category $(\mathcal{C}, \otimes, I)$, we define a *word* recursively.

(i) We have a stack of *variables* $A, B, C, \dots$, which are all words.

(ii) The unit I is a word.

(iii) If u, v are words, then $u \otimes v$ is a word.

A word with n variables defines a functor $\mathcal{C}^n \to \mathcal{C}$.

Theorem (Mac Lane's coherence theorem)**.** For any two words w, w' with the same sequence of variables in the same order, there is a unique natural isomorphism $w \to w'$ obtained by composing instances of α, λ, ρ and their inverses.

Proof. We define the *height* of a word w to be $a(w) + i(w)$, where

(i) $a(w)$ is the *associator height*, which is the number of closing parentheses occurring immediately before $\otimes$ in w;

(ii) $i(w)$ is the number of occurrences of I in w.

Applying any instance of α, λ, ρ to a word reduces its height. For example, if $\alpha \dots : w \to w'$, then $a(w') < a(w)$ and $i(w') = i(w)$, and correspondingly if $\lambda \dots w \to w'$, then $i(w') = i(w) - 1$ and $a(w') \leq a(w)$. In particular, any string of instances of α, λ, ρ starting from w has length at most $a(w) + i(w)$.

We say that a word w is *reduced* if either $a(w) = i(w) = 0$ or $w = I$. If $a(w) > 0$, then w is the domain of an instance of α, and if $i(w) > 0$ and $w \neq I$, then w is the domain of an instance of either λ or ρ. Thus, for any word w, there is a string $w \to \dots \to w_0$ where w_0 is the unique reduced word containing the same variables of w in the same order. We must show that any two such strings have the same composite. Given

$$\begin{tikzcd} & w \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ w' & & w'' \end{tikzcd}$$

where φ, ψ are instances of α, λ, or ρ, we need to find a word w''' completing the commutative square

$$\begin{tikzcd} & w \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ w' \arrow[dr, "\theta"'] & & w'' \arrow[dl, "\chi"] \\ & w''' & \end{tikzcd}$$

where θ, χ are composites of instances of α, λ, and ρ.

If φ, ψ act on disjoint subwords of w, so $w = u \otimes v$ where $\varphi = \varphi' \otimes 1_v$ and $\psi = 1_u \otimes \psi'$, then we can fill in the square as follows.

$$\begin{tikzcd} & u \otimes v \arrow[dl, "\varphi' \otimes 1_v"'] \arrow[dr, "1_u \otimes \psi'"] & \\ u' \otimes v \arrow[dr, "1_{u'} \otimes \psi'"'] & & u \otimes v' \arrow[dl, "\varphi' \otimes 1_{v'}"] \\ & u' \otimes v' & \end{tikzcd}$$

Now suppose one acts within the argument of the other, for example, if φ is $\alpha_{t,u,v}$ and $\psi = (1_t \otimes \psi') \otimes 1_v$. Then by naturality of α, we can complete the diagram with $1_t \otimes (\psi' \otimes 1_v)$ and $\alpha_{t,u',v}$.

Now suppose that φ and ψ interfere. If φ and ψ are both instances of α, then the pentagon equation completes the commutative square.

Suppose one is an instance of α and the other is an instance of λ or ρ. Then I must occur as one of the three arguments to α. If it is the middle argument, the two diagrams in the definition of a monoidal category complete the square. If if is the left or right argument, the other two diagrams defined immediately after will complete the square.

Finally, if one is an instance of λ and the other is an instance of ρ, then they must be λ_I and ρ_I, and so must agree. This completes the proof that there is a unique natural isomorphism to a reduced word.

Now suppose we have a string

$$\begin{tikzcd} w_1 \arrow[r] & w_2 & w_3 \arrow[l] \arrow[r] & w_4 & \cdots & w_n \end{tikzcd}$$

Then there are unique 'forwards' morphisms

$$\begin{tikzcd} w_1 \arrow[r] \arrow[drr] & w_2 \arrow[dr] & w_3 \arrow[l] \arrow[r] \arrow[d] & w_4 \arrow[dl] & \cdots & w_n \arrow[dlll] \\ & & w_0 & & & \end{tikzcd}$$

to w_0, which is the reduced word with the same sequence of variables. Each of the triangles must commute by the uniqueness result proven above. Hence the composite of the arrows along the top edge is equal to the composite $w_1 \to w_0 \leftarrow w_n$. $\square$

Definition. A *symmetry* on a monoidal category $(\mathcal{C}, \otimes, I)$ is a natural isomorphism $\gamma_{A,B}$: $A \otimes B \to B \otimes A$ such that the following diagrams commute.

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A\otimes B,C}"'] & A \otimes (B \otimes C) \arrow[r, "1_A \otimes \gamma_{B,C}"] & A \otimes (C \otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] \\ C \otimes (A \otimes B) & (C \otimes A) \otimes B \arrow[l, "\alpha_{C,A,B}"] & (A \otimes C) \otimes B \arrow[l, "\gamma_{A,C} \otimes 1_B"] \end{tikzcd}$$

$$\begin{tikzcd} A \otimes I \arrow[rr, "\gamma_{A,I}"] \arrow[dr, "\rho_A"'] & & I \otimes A \arrow[dl, "\lambda_A"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes B \arrow[r, "\gamma_{A,B}"] \arrow[dr, "1_{A\otimes B}"'] & B \otimes A \arrow[d, "\gamma_{B,A}"] \\ & A \otimes B \end{tikzcd}$$

For the weaker notion of a *braiding*, we can omit the last of the three diagrams, but add an additional hexagonal equation, since it can no longer be derived from the first.

There is a coherence theorem for symmetric monoidal categories, which is also due to Mac Lane. The theorem shows that for any two words w, w' involving the same set of variables without repetition, there is a unique natural isomorphism between w and w' obtained from compositions of instances of α, λ, γ and their inverses. Note that ρ is not necessary, as it can be produced from instances of λ and γ. The examples of monoidal categories above are all symmetric, except for (iv) and (v).

6.3 Monoidal functors

Definition. Let $(\mathcal{C}, \otimes, I), (\mathcal{D}, \oplus, J)$ be monoidal categories. A *(lax) monoidal functor* $F : (\mathcal{C}, \otimes, I) \to (\mathcal{D}, \oplus, J)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a natural transformation $\varphi_{A,B} : FA \oplus FB \to F(A \otimes B)$ and a morphism $\iota : J \to FI$, such that the following diagrams commute.

$$\begin{tikzcd}
(FA \oplus FB) \oplus FC \arrow[r, "\varphi_{A,B} \oplus 1_{FC}"] \arrow[d, "\alpha_{FA,FB,FC}"'] & F(A \otimes B) \oplus FC \arrow[r, "\varphi_{A\otimes B,C}"] & F((A \otimes B) \otimes C) \arrow[d, "F\alpha_{A,B,C}"] \\
FA \oplus (FB \oplus FC) \arrow[r, "1_{FA} \oplus \varphi_{B,C}"'] & FA \oplus F(B \otimes C) \arrow[r, "\varphi_{A,B\otimes C}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
J \oplus FA \arrow[r, "\iota \oplus 1_{FA}"] \arrow[d, "\lambda_{FA}"'] & FI \oplus FA \arrow[d, "\varphi_{I,A}"] \\
FA & F(I \otimes A) \arrow[l, "F\lambda_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \oplus J \arrow[r, "1_{FA} \oplus \iota"] \arrow[d, "\rho_{FA}"'] & FA \oplus FI \arrow[d, "\varphi_{A,I}"] \\
FA & F(A \otimes I) \arrow[l, "F\rho_A"]
\end{tikzcd}$$

We say F is *strong monoidal* (respectively *strict monoidal*) if φ and ι are isomorphisms (respectively identities). An *oplax* monoidal functor is the same definition, but where the directions of the maps φ and ι are reversed.

Note that the same letters are used for the associators and unitors in both monoidal categories.

Example. (i) The forgetful functor $U : (\mathbf{AbGp}, \otimes, \mathbb{Z}) \to (\mathbf{Set}, \times, 1)$ is lax monoidal. We define $\iota : 1 \to \mathbb{Z}$ to map the element of 1 to the generator $1 \in \mathbb{Z}$, and define $\varphi : UA \times UB \to U(A \otimes B)$ by $(a, b) \mapsto a \otimes b$. One can easily verify that the required diagrams commute.

(ii) The free functor $F : (\mathbf{Set}, \times, 1) \to (\mathbf{AbGp}, \otimes, \mathbb{Z})$ is strong monoidal, because $F1 \cong \mathbb{Z}$ and $F(A \times B) \cong FA \otimes FB$.

(iii) Let R be a commutative ring. Then the forgetful functor $\mathbf{Mod}_R \to \mathbf{AbGp}$ is lax monoidal, where $\iota : \mathbb{Z} \to R$ is the natural map, and $\varphi : A \otimes_{\mathbb{Z}} B \to A \otimes_R B$ is the quotient map. Its left adjoint, the free functor $\mathbf{AbGp} \to \mathbf{Mod}_R$, is strong monoidal.

(iv) If $\mathcal{C}$ and $\mathcal{D}$ have the cartesian monoidal structure, then any functor $F : \mathcal{C} \to \mathcal{D}$ is oplax monoidal. $\iota : F1 \to 1$ is the unique morphism to the terminal object of $\mathcal{D}$, and $\varphi_{A,B} : F(A \times B) \to FA \times FB$ is given by $(F\pi_1, F\pi_2)$. F is strong monoidal if and only if it preserves finite products.

(v) If X and Y are metric spaces, then $1_{X \times Y}$ is non-expansive as a map $(X \times Y, d_1) \to (X \times Y, d_\infty)$, making the identity functor $1_{\mathbf{Met}}$ into a monoidal functor $(\mathbf{Met}, \times_\infty, 1) \to (\mathbf{Met}, \times_1, 1)$. Note that the d_∞ metric on $X \times Y$ defines the categorical product.

Lemma. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Let $F \dashv G$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then there is a bijection between lax monoidal structures on G and oplax monoidal structures on F.

Proof sketch. Suppose we have (φ, ι) on G. Then the transpose of $\iota : J \to GI$ is a morphism $FJ \to I$, and we have a natural transformation

$$F(A \otimes B) \xrightarrow{F(\eta_A \times \eta_B)} F(GFA \otimes GFB) \xrightarrow{F\varphi_{FA,FB}} FG(FA \oplus FB) \xrightarrow{\epsilon_{FA \oplus FB}} FA \oplus FB$$

One can check that each of the required diagrams commute, defining an oplax monoidal structure on F. By duality, an oplax monoidal structure on F yields a lax monoidal structure on G, and it can be shown that these constructions are inverse to each other. □

6.4 Closed monoidal categories

Definition. We say that a monoidal category $(\mathcal{C}, \otimes, I)$ is *(left/right/bi)-closed* if $A \otimes (-)$, $(-) \otimes A$, or both have right adjoints for all A. If $\otimes$ is symmetric, we say in any of these cases that $\mathcal{C}$ is *closed*.

Right adjoints for $(-) \otimes A$ are denoted $[A, -]$ if they exist.

Example. (i) A cartesian closed category is a monoidal category with $\otimes = \times$, that is closed as a monoidal category. In particular, **Set** and **Cat** are cartesian closed.

(ii) The metric d_1 on the set $[X, Y]$ of non-expansive maps $X \to Y$ yields a closed structure on $(\mathbf{Met}, \times_1, 1)$.

(iii) **AbGp** and $\mathbf{Mod}_R$ for any commutative ring R are monoidal closed, where $[A, B]$ is the set of homomorphisms $A \to B$, turned into an abelian group or R-module by pointwise addition and scalar multiplication. The homomorphisms $C \to [A, B]$ correspond under λ-conversion to bilinear maps $C \times A \to B$, and thus to homomorphisms $C \otimes_R A \to B$.

(iv) The cartesian monoidal structure on the category of pointed sets $\mathbf{Set}_\star$ is not closed, but the monoidal structure given by the *smash product* $(-) \wedge (-)$ is closed, where

$$(A, a_0) \wedge (B, b_0) = {}^{A \times B}\!/\!_{\sim}$$

and $\sim$ identifies all elements where either coordinate is the basepoint. Basepoint-preserving maps $A \wedge B \to C$ correspond to basepoint-preserving maps from A to the set $[B, C]$ of basepoint-preserving maps $B \to C$.

(v) Consider the set $\mathrm{Rel}(A \times A) = P(A \times A)$ of relations on A. This is a poset under inclusion, and is a monoid under relational composition. Composition is order-preserving in each variable, making $\mathrm{Rel}(A \times A)$ into a strict monoidal category. It is not symmetric, but biclosed. For the right adjoint to $(-) \circ R$, we define $R \Rightarrow T$ to be

$$(R \Rightarrow T) = \{(b, c) \in A \times A \mid \forall a \in A,\ (a, b) \in R \Rightarrow (a, c) \in T\}$$

Then $S \subseteq (R \Rightarrow T)$ if and only if $S \circ R \subseteq T$.

6.5 Enriched categories

Definition. Let $(\mathcal{E}, \otimes, I)$ be a monoidal category. An $\mathcal{E}$*-enriched category* consists of

(i) a collection ob $\mathcal{C}$ of objects;

(ii) an object $\mathcal{C}(A, B)$ of $\mathcal{E}$ for each pair of objects $A, B \in \mathrm{ob}\,\mathcal{C}$;

(iii) morphisms $\iota_A : I \to \mathcal{C}(A,A)$ for each A;
(iv) morphisms $\kappa_{A,B,C} : \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \to \mathcal{C}(A,C)$ for objects A, B, C,

such that the following diagrams commute.

$$\begin{tikzcd}
I \otimes \mathcal{C}(A,B) \arrow[r, "\iota_B \otimes 1_{\mathcal{C}(A,B)}"] \arrow[dr, "\lambda_{\mathcal{C}(A,B)}"'] & \mathcal{C}(B,B) \otimes \mathcal{C}(A,B) \arrow[d, "\kappa_{A,B,B}"] \\
& \mathcal{C}(A,B)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(A,B) \otimes I \arrow[r, "\iota_B \otimes 1_{\mathcal{C}(A,B)}"] \arrow[dr, "\rho_{\mathcal{C}(A,B)}"'] & \mathcal{C}(A,B) \otimes \mathcal{C}(A,A) \arrow[d, "\kappa_{A,A,B}"] \\
& \mathcal{C}(A,B)
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{C}(C,D) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(A,B) \arrow[r, "\kappa \otimes 1"] \arrow[dd, "\alpha"'] & \mathcal{C}(B,D) \otimes \mathcal{C}(A,B) \arrow[dr, "\kappa"] & \\
& & \mathcal{C}(A,D) \\
\mathcal{C}(C,D) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \arrow[r, "1 \otimes \kappa"'] & \mathcal{C}(C,D) \otimes \mathcal{C}(A,C) \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

Definition. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{E}$-enriched categories. An *$\mathcal{E}$-enriched functor* $\mathcal{C} \to \mathcal{D}$ consists of a map of objects $F : \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$ together with morphisms $F_{A,B} : \mathcal{C}(A,B) \to \mathcal{D}(FA, FB)$ for each pair of objects $A, B \in \operatorname{ob} \mathcal{C}$, in such a way that is compatible with identities and composition.

Definition. Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be $\mathcal{E}$-enriched functors between $\mathcal{E}$-enriched categories. An *$\mathcal{E}$-enriched natural transformation* $F \to G$ assigns a morphism $\theta_A : I \to \mathcal{D}(FA, GA)$ to each $A \in \operatorname{ob} \mathcal{C}$, satisfying the naturality condition

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[r, "F_{A,B}"] \arrow[d, "G_{A,B}"'] & \mathcal{D}(FA,FB) \arrow[r, "\lambda^{-1}"] & I \otimes D(FA,FB) \arrow[d, "\theta_B \otimes 1"] \\
\mathcal{D}(GA,GB) \arrow[d, "\rho^{-1}"'] & & \mathcal{D}(FB,GB) \otimes \mathcal{D}(FA,FB) \arrow[d, "\kappa"] \\
\mathcal{D}(GA,GB) \otimes I \arrow[r, "1 \otimes \theta_A"'] & \mathcal{D}(GA,GB) \otimes \mathcal{D}(FA,GA) \arrow[r, "\kappa"'] & \mathcal{D}(FA,GB)
\end{tikzcd}$$

If $\mathcal{C}$ is an $\mathcal{E}$-enriched category, its *underlying ordinary category* $|\mathcal{C}|$ is the category where the objects are those of $\mathcal{C}$, the morphisms $A \to B$ are the morphisms $I \to \mathcal{C}(A,B)$ in $\mathcal{E}$, where the identity morphisms are given by ι_A, and the composition of $g : C \to B$ and $f : A \to B$ given by

$$I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{g \otimes f} \mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \xrightarrow{\kappa} \mathcal{C}(A,C)$$

One can check that this indeed forms a category. An *$\mathcal{E}$-enrichment* of an ordinary category $\mathcal{C}_0$ is an $\mathcal{E}$-enriched category $\mathcal{C}$ such that $|\mathcal{C}| \cong \mathcal{C}_0$.

Example. (i) A category enriched over $(\mathbf{Set}, \times, 1)$ is a locally small category.

(ii) A category enriched over the poset $2 = \{0, 1\}$ with $0 < 1$ is a preorder.

(iii) A category enriched over $(\mathbf{Cat}, \times, \mathbf{1})$ is a *2-category*. Its morphisms or *1-arrows* $A \to B$ are the objects of a category $\mathcal{C}(A, B)$. It has *2-arrows* between parallel pairs $f, g : A \rightrightarrows B$, which are the morphisms $f \to g$ in the category $\mathcal{C}(A, B)$. **Cat** is a 2-category, by taking the 2-arrows to be the natural transformations. The category of small $\mathcal{E}$-enriched categories with $\mathcal{E}$-enriched functors is a 2-category.

(iv) A category enriched over $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ is an *additive category*.

(v) If $\mathcal{E}$ is a right closed monoidal category, it has a canonical enrichment structure over itself. Take $\mathcal{E}(A, B)$ to be $[A, B]$, where $[A, -]$ is the right adjoint of $(-) \otimes A$. The identity $I \to [A, A]$ is the transpose $\lambda_A : I \otimes A \to A$, and the composition κ is the transpose of

$$([B, C] \otimes [A, B]) \otimes A \xrightarrow{\alpha} [B, C] \otimes ([A, B] \otimes A) \xrightarrow{1 \otimes \mathrm{ev}} [B, C] \otimes B \xrightarrow{\mathrm{ev}} C$$

where ev is the evaluation map, which is precisely the counit of the adjunction.

(vi) A one-object $\mathcal{E}$-enriched category is an *(internal) monoid* in $\mathcal{E}$; it consists of an object M of $\mathcal{E}$, equipped with morphisms $e : I \to M$ and $m : M \otimes M \to M$ satisfying the left and right unit laws and the associativity law.

(a) An internal monoid in **Set** is a monoid.

(b) An internal monoid in **AbGp** is a ring.

(c) An internal monoid in **Cat** is a strict monoidal category.

(d) An internal monoid in $[\mathcal{C}, \mathcal{C}]$ is a monad on $\mathcal{C}$.

7 Additive and abelian categories

7.1 Additive categories

In this section, we will study categories enriched over $(\mathbf{AbGp}, \otimes, \mathbb{Z})$; these are called *additive* categories. We will also consider other weaker enrichments: a category enriched over $(\mathbf{Set}_\star, \wedge, 2)$ is called *pointed*, and a category enriched over $(\mathbf{CMon}, \otimes, \mathbb{N})$, where **CMon** is the category of commutative monoids, is called *semi-additive*.

In a pointed category $\mathcal{C}$, each $\mathcal{C}(A, B)$ has a distinguished element 0, and all composites with zero morphisms are zero morphisms. In a semi-additive category $\mathcal{C}$, each $\mathcal{C}(A, B)$ has a binary addition operation which is associative, commutative, and has an identity 0. Composition in a semi-additive category is bilinear, so $(f + g)(h + k) = fh + gh + fk + gk$ whenever the composites are defined. In an additive category, each morphism $f \in \mathcal{C}(A, B)$ has an additive inverse $-f \in \mathcal{C}(A, B)$.

Lemma. (i) For an object A in a pointed category $\mathcal{C}$, the following are equivalent.
(a) A is a terminal object of $\mathcal{C}$.
(b) A is an initial object of $\mathcal{C}$.

(c) $1_A = 0 : A \to A$.

(ii) For objects A, B, C in a semi-additive category $\mathcal{C}$, the following are equivalent.

(a) there exist morphisms $\pi_1 : C \to A$ and $\pi_2 : C \to B$ making C into a product of A and B;

(b) there exist morphisms $\nu_1 : A \to C$ and $\nu_2 : B \to C$ making C into a coproduct of A and B;

(c) there exist morphisms $\pi_1 : C \to A, \pi_2 : C \to B, \nu_1 : A \to C, \nu_2 : B \to C$ satisfying

$$\pi_1\nu_1 = 1_A; \quad \pi_2\nu_2 = 1_B; \quad \pi_1\nu_2 = 0; \quad \pi_2\nu_1 = 0; \quad \nu_1\pi_1 + \nu_2\pi_1 = 1_C$$

Proof. In each part, as (a) and (b) are dual and (c) is self-dual, it suffices to prove the equivalence of (a) and (c).

Part (i). If A is terminal, then it has exactly one morphism $A \to A$, so this must be the zero morphism. Conversely, if $1_A = 0$, then A is terminal, as for any $f : B \to A$, we have $f = 1_A f = 0f = 0$, so the only morphism $B \to A$ is the zero morphism.

Part (ii). If (a) holds, take ν_1, ν_2 to be defined by the first four equations in (c); it suffices to verify the last equation, $\nu_1\pi_1 + \nu_2\pi_2 = 1_C$. Composing with π_1,

$$\pi_1\nu_1\pi_1 = 1_A\pi_1 + 0\pi_2 = \pi_1$$

and similarly, composing with π_2 gives π_2. So by uniqueness of factorisations through limit cones, $\nu_1\pi_1 + \nu_2\pi_2$ must be the identity. Conversely, if (c) holds, given a pair $f : D \to A$ and $g : D \to B$, the morphism

$$h = \nu_1 f + \nu_2 g$$

satisfies

$$\pi_1 h = 1_A f + 0g = f; \quad \pi_2 h = 0f + 1_A g = g$$

giving a factorisation, and if h' also satisfies these equations, then

$$h' = (\nu_1\pi_1 + \nu_2\pi_2)h' = \nu_1 f + \nu_2 g = h$$

so the factorisation is unique. □

In any category, an object which is both initial and terminal is called a *zero object*, denoted 0. An object that is a product and a coproduct of A and B is called a *biproduct*, denoted $A \oplus B$.

Lemma. Let $\mathcal{C}$ be a locally small category.

(i) If $\mathcal{C}$ has a zero object, then it has a unique pointed structure.

(ii) Suppose $\mathcal{C}$ has a zero object and has binary products and coproducts. Suppose further that for each pair $A, B \in \operatorname{ob}\mathcal{C}$, the canonical morphism $c : A + B \to A \times B$ defined by

$$\pi_i c \nu_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is an isomorphism. Then $\mathcal{C}$ has a unique semi-additive structure.

We adopt the convention that morphisms into a product are denoted with column vectors, and morphisms out of a product are denoted with row vectors.

Proof. *Part (i).* The unique morphism $0 \to 0$ is both the identity and a zero morphism. So for any two $A, B : \mathrm{ob}\,\mathcal{C}$, the unique composite $A \to 0 \to B$ must be the zero element of $\mathcal{C}(A, B)$. We can define a pointed structure on $\mathcal{C}$ in this way.

Part (ii). This technique is known as the *Eckmann–Hilton argument*. Given $f, g : A \rightrightarrows B$, we define the *left sum* $f +_\ell g$ to be the composite

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} B$$

and the *right sum* $f +_r g$ to be

$$A \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} A \times A \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} B$$

Note that $(f +_\ell g)h = fh +_\ell gh$, since

$$\begin{pmatrix} f \\ g \end{pmatrix} h = \begin{pmatrix} fh \\ gh \end{pmatrix}$$

and similarly,

$$k(f +_r g) = kf +_r kg$$

So if we show that the two sums coincide, we obtain the required distributive laws. First, note that $0 : A \to B$ is a two-sided identity for both $+_\ell$ and $+_r$. For example, $f +_\ell 0 = f$, since

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, swap, "\begin{pmatrix} f \\ 0 \end{pmatrix}"] & & B \arrow[rr, "1_B"] \arrow[dl, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[dr, "\nu_1"] & & B \\
& B \times B \arrow[rr, swap, "c^{-1}"] & & B + B \arrow[ur, swap, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] &
\end{tikzcd}$$

commutes. Suppose we have morphisms $f, g, h, k : A \to B$, and consider the composite

$$A \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} A \times A \xrightarrow{c^{-1}} A + A \xrightarrow{\begin{pmatrix} f & g \\ h & k \end{pmatrix}} B \times B \xrightarrow{c^{-1}} B + B \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} B$$

The composite of the first three factors is

$$\begin{pmatrix} f +_r g \\ h +_r k \end{pmatrix}$$

so the whole composite is $(f +_r g) +_\ell (h +_r k)$. Evaluating from other end, we obtain

$$(f +_r g) +_\ell (h +_r k) = (f +_\ell h) +_r (g +_\ell k)$$

This is known as the *interchange law*. Substituting $g = k = 0$, we obtain $f +_\ell k = f +_r k$. Substituting $f = k = 0$ (and dropping the subscripts) we obtain the commutative law $g + h = h + g$. Substituting $h = 0$, we obtain the associativity law $(f + g) + k = f + (g + k)$.

For uniqueness, suppose we have some semi-additive structure $+$ on $\mathcal{C}$. Then $\nu_1\pi_1 + \nu_2\pi_2$ must be the inverse of $c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : A + B \to A \times B$, since

$$\nu_1\pi_1 c = \nu_1 \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} \nu_1 & 0 \end{pmatrix}; \quad \nu_2\pi_2 c = \begin{pmatrix} 0 & \nu_2 \end{pmatrix}$$

so

$$(\nu_1\pi_1 + \nu_2\pi_2)c = \begin{pmatrix}\nu_1 + 0 & 0 + \nu_2\end{pmatrix} = \begin{pmatrix}\nu_1 & \nu_2\end{pmatrix} = 1_{A+B}$$

Hence the definitions of $+_\ell$ and $+_r$ both reduce to $+$. □

Note that if $\mathcal{C}$ and $\mathcal{D}$ are semi-additive categories with finite biproducts, then a functor $F : \mathcal{C} \to \mathcal{D}$ is semi-additive (that is, enriched over **CMon**) if and only if it preserves either finite products or finite coproducts. In particular, if F has either a left or right adjoint, then it is semi-additive, and the adjunction is enriched over **CMon**; the bijection $\mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$ is an isomorphism of commutative monoids, since the operations $F(-)$ and $(-)\epsilon_B$ both respect addition.

7.2 Kernels and cokernels

Definition. Let $f : A \to B$ be a morphism in a pointed category $\mathcal{C}$. The *kernel* of f is the equaliser of the pair $(f, 0)$; dually the *cokernel* is the coequaliser of $(f, 0)$. A monomorphism that occurs as the kernel of a morphism is called *normal*.

In an additive category, the normal monomorphisms are precisely the regular monomorphisms, since the equaliser of (f, g) is the kernel of $f - g$. In **Gp**, all inclusions of subgroups are regular, but not all inclusions are normal, since a normal monomorphism corresponds to a normal subgroup. In $\mathbf{Set}_\star$, all surjections are regular epimorphisms, but $(A, a_0) \to (B, b_0)$ is a normal epimorphism if f is bijective on elements not mapped to b_0. We say that a morphism $f : A \to B$ is a *pseudomonomorphism* if its kernel is a zero morphism; that is, $fg = 0$ implies $g = 0$.

Lemma. In a pointed category with kernels and cokernels, $f : A \to B$ is normal monic if and only if $f \cong \ker \operatorname{coker} f$.

Proof. If $f \cong \ker \operatorname{coker} f$, it is clearly normal. Now suppose $f = \ker g$. Then g factors through the cokernel of f, so $g(\ker \operatorname{coker} f) = 0$. Thus $\ker \operatorname{coker} f \le f$ in $\mathrm{Sub}(B)$. But $(\operatorname{coker} f)f = 0$, so $f \le \ker \operatorname{coker} f$, so they are isomorphic as subobjects of B. □

Corollary. In a pointed category with kernels and cokernels, the operations ker and coker induce an order-reversing bijection between isomorphism classes of normal subobjects and isomorphism classes of normal quotients of any object.

Remark. For any morphism $f : A \to B$ in such a category, $\ker \operatorname{coker} f$ is the smallest normal subobject of B through which f factors.

7.3 Abelian categories

Definition. An *abelian category* is an additive category with all finite limits and colimits. Equivalently, an abelian category is a category with a zero object, finite biproducts, kernels, and cokernels, such that all monomorphisms and epimorphisms are normal.

Example. (i) The category **AbGp** is abelian; more generally, for any ring R, the category $\mathbf{Mod}_R$ is abelian.

(ii) If $\mathcal{A}$ is abelian and $\mathcal{C}$ is small, then $[\mathcal{C}, \mathcal{A}]$ is abelian, with all structures defined pointwise.

(iii) If $\mathcal{A}$ is abelian and $\mathcal{C}$ is small and additive, then the category of additive functors $\mathcal{C} \to \mathcal{A}$, denoted $\mathrm{Add}(\mathcal{C}, \mathcal{A})$, is also abelian, as it is closed under all of the structures on $[\mathcal{C}, \mathcal{A}]$. Note that this covers the case of R-modules, as an additive category with a single object is a ring, and the category of modules over such a ring is isomorphic to the category of additive functors from this category to **AbGp**.

Remark. If $f : A \to B$ in an abelian category, then $\ker \operatorname{coker} f$ is the smallest subobject $I \rightarrowtail B$ through which f factors. This is called the *image* of f, denoted $\operatorname{im} f = \ker \operatorname{coker} f$. The other part of the factorisation $A \to I$ is epic, as it cannot factor through the equaliser of any nonequal parallel pair $I \rightrightarrows C$. Thus, it is also the smallest quotient of A through which f factors, so it is the *coimage* of f, given by $\operatorname{coim} f = \operatorname{coker} \ker f$. The composition $A \twoheadrightarrow I \rightarrowtail B$ is the unique epi–mono factorisation of f.

To show that this factorisation is stable under pullback, it suffices to show that the pullback of an epimorphism in an abelian category is epic, as the corresponding statement for monomorphisms has already been shown.

Lemma (flattening lemma)**.** Consider a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

in an abelian category $\mathcal{A}$. Its *flattening* is the sequence

$$A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} h & -k \end{pmatrix}} D$$

Then

(i) the square commutes if and only if the composite of the flattening $\begin{pmatrix} h & -k \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$ is the zero morphism;

(ii) the square is a pullback if and only if $\begin{pmatrix} f \\ g \end{pmatrix} = \ker \begin{pmatrix} h & -k \end{pmatrix}$;

(iii) the square is a pushout if and only if $\begin{pmatrix} h & -k \end{pmatrix} = \operatorname{coker} \begin{pmatrix} f \\ g \end{pmatrix}$.

Proof. Part (i). The composite $\begin{pmatrix} h & -k \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$ is $hf - kg$, so it vanishes if and only if the square commutes.

Part (ii). $\begin{pmatrix} f \\ g \end{pmatrix}$ is the kernel of $\begin{pmatrix} h & -k \end{pmatrix}$ if and only if

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

is universal among spans completing the cospan

$$\begin{tikzcd} & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

into a commutative square.

Part (iii). Follows by duality, taking care of the asymmetric negation. □

Corollary. In an abelian category $\mathcal{A}$, epimorphisms are stable under pullback.

Proof. Suppose we have a pullback square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, two heads, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

By part (ii) of the above result, $\begin{pmatrix} f \\ g \end{pmatrix} = \ker\begin{pmatrix} h & -k \end{pmatrix}$. But h is an epimorphism, so $\begin{pmatrix} h & -k \end{pmatrix}$ is also an epimorphism. Thus $\begin{pmatrix} h & -k \end{pmatrix} = \operatorname{coker}\begin{pmatrix} f \\ g \end{pmatrix}$, so the square is also a pushout. We show that g is a pseudoepimorphism; this suffices as $\mathcal{A}$ is additive. Suppose we have $\ell : C \to E$ with $\ell g = 0$. Then $\begin{pmatrix} \ell & (B \xrightarrow{0} E) \end{pmatrix}$ factors uniquely through the pushout.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, two heads, "h"] \arrow[ddr, bend left, "0"] & \\ C \arrow[r, "k"'] \arrow[drr, bend right, "\ell"'] & D \arrow[dr, "m"] & \\ & & E \end{tikzcd}$$

But then $mh = 0$ and h is epic, so $m = 0$, giving $\ell = mk = 0$. □

Thus image factorisations are stable under pullback, and dually, under pushout.

7.4 Exact sequences

Definition. A sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$$

in an abelian category $\mathcal{A}$ is *exact* at A_n if $\ker f_n = \operatorname{im} f_{n+1}$. The entire sequence is said to be *exact* if it is exact at every vertex.

By duality, the sequence is exact at A_n if and only if $\operatorname{coker} f_{n+1} = \operatorname{coim} f_n$.

Example.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at A if and only if f is monic, and is exact at A and B if and only if $f = \ker g$.

Definition. A functor between abelian categories $F : \mathcal{A} \to \mathcal{B}$ is *exact* if it preserves arbitrary exact sequences.

This implies that F preserves kernels and cokernels, and the converse is true as images are defined in terms of kernels and cokernels.

Definition. F is *left exact* if it preserves exact sequences of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

Proposition. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. Then
(i) F is left exact if and only if it preserves all finite limits (and hence is additive);
(ii) F is exact if and only if it is left exact and preserves epimorphisms.

Proof. Part (i). One direction is trivial as kernels are finite limits. Conversely, note that for any A, B, the sequence

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} B \longrightarrow 0$$

is exact, and conversely, if we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

and either f is a split monomorphism or g is a split epimorphism, then $C \cong A \oplus B$. Indeed, suppose that f is split, so $rf = 1_A$. Then $g = \operatorname{coker} f = \operatorname{coker} fr$ is the equaliser of $(1_C - fr, 1_C)$, so it is the epic part of a splitting of the idempotent $1_C - fr$. If $s : B \to C$ is the monic part of this splitting, then the four morphisms (r, g, f, s) satisfy the equations of a biproduct. So F maps

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} B \longrightarrow 0$$

to a sequence identifying $F(A \oplus B)$ as $FA \oplus FB$, and thus preserves biproducts. Hence F preserves all finite limits.

Part (ii). If F is left exact and preserves epimorphisms, then it preserves the exactness of sequences of the form

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

Thus it preserves kernels and cokernels. □

7.5 The five lemma

Lemma. Suppose we have a commutative diagram in an abelian category

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "u_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "u_4"] & A_5 \arrow[d, "u_5"] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \end{tikzcd}$$

where the rows are exact sequences. Then,

(i) if u_1 is epic and u_2, u_4 are monic, then u_3 is monic;

(ii) if u_5 is monic and u_2, u_4 are spic, then u_3 is epic.

Thus if u_1, u_2, u_4, u_5 are isomorphisms, u_3 is an isomorphism.

Proof. By duality it suffices to show (i). We show u_3 is a pseudomonomorphism. Suppose we have $x : C \to A_3$ with $u_3 x = 0$. Then $u_4 f_3 x = g_4 u_3 x = 0$, so as u_4 is a monomorphism, $f_3 x = 0$. Hence x factors through the kernel of f_3, which is the image of f_2. Form the pullback of f_2 and x to obtain

$$\begin{tikzcd} & D \arrow[r, "y"] \arrow[d, "z"] & C \arrow[d, "x"] & & \\ A_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "u_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "u_4"] & A_5 \arrow[d, "u_5"] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \end{tikzcd}$$

Then y is also the pullback of this factorisation of x along coim f_2, so y is an epimorphism as epimorphisms are stable under pullback. Then $g_2 u_2 z = u_3 f_2 z = u_3 x y = 0$. Thus $u_2 z$ factors through $\ker g_2 = \operatorname{im} g_1$. Consider the pullback square

$$\begin{tikzcd} E \arrow[r, "v"] \arrow[d, "w"'] & D \arrow[d, "u_2 z"] \\ A_1 \arrow[r, "g_1 u_1"'] & B_2 \end{tikzcd}$$

So v is epic, as it is the pullback of $\operatorname{coim}(g_1 u_1)$.

$$\begin{tikzcd}
E \arrow[r, "v"] \arrow[d, "w"] & D \arrow[r, "y"] \arrow[d, "z"] & C \arrow[d, "x"] & & \\
A_1 \arrow[r, "f_1"] \arrow[d, "u_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "u_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "u_3"] & A_4 \arrow[r, "f_4"] \arrow[d, "u_4"] & A_5 \arrow[d, "u_5"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5
\end{tikzcd}$$

Thus $u_2 z v = g_1 u_1 w$, and u_2 is monic, so $zv = f_1 w$. Then $xyv = f_2 z v = f_2 f_1 w = 0$, and yv is epic, hence $x = 0$. $\square$

7.6 The snake lemma

Lemma. Consider a diagram in an abelian category

$$\begin{tikzcd}
& B_1 \arrow[r] \arrow[d, "v_1"] & B_2 \arrow[r] \arrow[d, "v_2"] & B_3 \arrow[r] \arrow[d, "v_3"] & 0 \\
0 \arrow[r] & C_1 \arrow[r] & C_2 \arrow[r] & C_3 &
\end{tikzcd}$$

where the rows are exact and the squares commute. Then we obtain an exact sequence

$$\begin{tikzcd}
& \operatorname{Ker} v_1 \arrow[r] \arrow[d] & \operatorname{Ker} v_2 \arrow[r] \arrow[d] & \operatorname{Ker} v_3 \arrow[d] \arrow[dddll, rounded corners, "s"] & \\
& B_1 \arrow[r] \arrow[d, "v_1"] & B_2 \arrow[r] \arrow[d, "v_2"] & B_3 \arrow[r] \arrow[d, "v_3"] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[d] & \\
& \operatorname{Coker} v_1 \arrow[r] & \operatorname{Coker} v_2 \arrow[r] & \operatorname{Coker} v_3 &
\end{tikzcd}$$

7.7 Complexes in abelian categories

Definition. Let $\mathcal{A}$ be an abelian category. A *(chain) complex* in $\mathcal{A}$ is an infinite sequence of objects and morphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

where the composite of any two consecutive morphisms is zero.

Note that a complex may be identified with an additive functor $\mathcal{Z} \to \mathcal{A}$, where $\mathcal{Z}$ is the additive category with $\operatorname{ob} \mathcal{Z} = \mathbb{Z}$ and

$$\mathcal{Z}(n,m) = \begin{cases} \mathbb{Z} & \text{if } m = n \text{ or } m = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, complexes on $\mathcal{A}$ are the objects of an abelian category $\mathsf{C}\mathcal{A} = \operatorname{Add}(\mathcal{Z}, \mathcal{A})$, where the morphisms are natural transformations.

Definition. Let $C_\bullet$ be a complex. We define
(i) $Z_n(C_\bullet) \rightarrowtail C_n$ to be the kernel of d_n;
(ii) $I_n(C_\bullet) \rightarrowtail C_n$ to be the image of d_{n+1};
(iii) $Z_n(C_\bullet) \twoheadrightarrow H_n(C_\bullet)$ to be the cokernel of $I_n(C_\bullet) \rightarrowtail Z_n(C_\bullet)$.
We say that $H_n(C_\bullet)$ is the *nth homology object* of $C_\bullet$.

Note that Z_n, I_n, H_n are additive functors $\mathsf{C}\mathcal{A} \to \mathcal{A}$.

Lemma. The construction of $H_n(C_\bullet)$ is self-dual.

Proof. Write $C_n \twoheadrightarrow Q_n(C_\bullet)$ for the cokernel of d_{n+1}. Then we have the diagram

$$\begin{tikzcd}
C_{n+1} \arrow[rrr, "d_{n+1}"] \arrow[d, two heads] & & & C_n \arrow[rr, "d_n"] \arrow[dr, two heads] & & C_{n-1} \\
I_n \arrow[r, tail] & Z_n \arrow[r, two heads] \arrow[urr, tail] & H_n & & Q_n \arrow[r, two heads] & I_{n-1} \arrow[u, tail]
\end{tikzcd}$$

By definition, $I_n \to C_n$ is $\ker(C_n \to Q_n)$. As $Z_n \to C_n$ is a monomorphism, $I_n \to Z_n$ is $\ker(Z_n \to C_n \to Q_n)$. Hence $Z_n \to H_n$ is $\operatorname{coim}(Z_n \to Q_n)$, so we obtain

$$\begin{tikzcd}
C_{n+1} \arrow[rrr, "d_{n+1}"] \arrow[d, two heads] & & & C_n \arrow[rr, "d_n"] \arrow[dr, two heads] & & C_{n-1} \\
I_n \arrow[r, tail] & Z_n \arrow[r, two heads] \arrow[urr, tail] & H_n \arrow[rr, tail] & & Q_n \arrow[r, two heads] & I_{n-1} \arrow[u, tail]
\end{tikzcd}$$

and $Z_n \twoheadrightarrow H_n \rightarrowtail Q_n$ is the image factorisation of $Z_n \to Q_n$. □

Theorem (Mayer–Vietoris sequence). Suppose we have a short exact sequence of complexes in $\mathcal{A}$.

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

Then there is a long exact sequence of homology objects

$$\cdots \longrightarrow H_n(A_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(B_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(C_\bullet) \longrightarrow H_{n-1}(A_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} H_{n-1}(B_\bullet) \xrightarrow{H_{n-1}(g_\bullet)} H_{n-1}(C_\bullet) \longrightarrow \cdots$$

Proof. First, we apply the snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"'] & B_n \arrow[r, "g_n"'] & C_n \arrow[r] & 0
\end{tikzcd}$$

to obtain exact sequences

$$0 \longrightarrow Z_{n+1}(A_\bullet) \longrightarrow Z_{n+1}(B_\bullet) \longrightarrow Z_{n+1}(C_\bullet)$$

and

$$Q_n(A_\bullet) \longrightarrow Q_n(B_\bullet) \longrightarrow Q_n(C_\bullet) \longrightarrow 0$$

Thus Z_n is a left exact functor and Q_n is right exact. We now apply the snake lemma again to the diagram

$$\begin{tikzcd}
 & Q_{n+1}(A_\bullet) \arrow[r] \arrow[d] & Q_{n+1}(B_\bullet) \arrow[r] \arrow[d] & Q_{n+1}(C_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_n(A_\bullet) \arrow[r] & Z_n(B_\bullet) \arrow[r] & Z_n(C_\bullet) &
\end{tikzcd}$$

Here, the cokernel of $Q_{n+1} \to Z_n$ coincides with that of $I_n \to Z_n$ as $Q_{n+1} \to I_n$ is epic. Their kernels coincide with $H_{n+1} \to Q_{n+1}$ as homology is self-dual. Hence we obtain

$$H_{n+1}(A_\bullet) \longrightarrow H_{n+1}(B_\bullet) \longrightarrow H_{n+1}(C_\bullet) \longrightarrow H_n(A_\bullet) \longrightarrow H_n(B_\bullet) \longrightarrow H_n(C_\bullet)$$

as required. □

Note that $Z_n : \mathrm{C}\mathcal{A} \to \mathcal{A}$ is the right adjoint to the functor $A \mapsto A[n]$, where $A[n]$ is the complex that has A in dimension n and 0 everywhere else; this gives another proof that Z is left exact. Dually, Q_n is the left adjoint to this functor.

Definition. Let $f_\bullet, g_\bullet : C_\bullet \rightrightarrows D_\bullet$ be two morphisms of $\mathrm{C}\mathcal{A}$. A *homotopy* from $f_\bullet$ to $g_\bullet$ is a sequence of morphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d_{n+1}h_n + h_{n-1}d_n$$

for all n. We say that $f_\bullet, g_\bullet$ are *homotopic* and write $f_\bullet \simeq g_\bullet$ if there exists such a sequence $h_\bullet$.

Homotopy is an equivalence relation on morphisms of $\mathrm{C}\mathcal{A}$. It is a congruence, as it is compatible with composition on both sides; indeed, if $k_\bullet : D_\bullet \to E_\bullet$, and $h_\bullet : f_\bullet \simeq g_\bullet$, then the morphisms $k_{n+1}h_n$ form a homotopy $k_\bullet f_\bullet \to k_\bullet g_\bullet$, and similarly for the other side. We write $\mathrm{H}\mathcal{A}$ for the quotient of $\mathrm{C}\mathcal{A}$ by the homotopy congruence. Also, homotopy is compatible with addition, by adding the relevant homotopies, so the quotient category inherits an additive structure, and the quotient $\mathrm{C}\mathcal{A} \to \mathrm{H}\mathcal{A}$ is an additive functor. In particular, $\mathrm{H}\mathcal{A}$ has finite biproducts, although it is not an abelian category.

Lemma. If $f_\bullet \simeq g_\bullet : C_\bullet \rightrightarrows D_\bullet$, then $H_n(f_\bullet) = H_n(g_\bullet)$ for all n.

Thus, the H_n can be regarded as additive functors $\mathrm{H}\mathcal{A} \to \mathcal{A}$.

Proof. Let $h_\bullet$ be a homotopy from $f_\bullet$ to $g_\bullet$, so $g_n - f_n = d_{n+1}h_n + h_{n-1}d_n$. Then $Z_n(g_\bullet) - Z_n(f_\bullet)$ is the restriction of $d_{n+1}h_n$ to $Z_n(C_\bullet)$, since $h_{n-1}d_n$ is zero on this subobject. Similarly, $H_n(g_\bullet) - H_n(f_\bullet)$ is zero, as $d_{n+1}h_n$ vanishes when factoring through the quotient. □

7.8 Projective resolutions

Definition. A category $\mathcal{C}$ has *enough projectives* if for every object A, there exists an epimorphism $P \twoheadrightarrow A$ where P is projective.

Note that this holds in **AbGp** and $\mathbf{Mod}_R$ for any commutative ring R, because free modules are projective, and every module can be written as a quotient of a free module.

Definition. Let $\mathcal{A}$ be an abelian category and let A be an object of $\mathcal{A}$. A *projective resolution* of A is a complex $P_\bullet$ where the objects P_n are projective, $P_n = 0$ for all $n < 0$, and

$$H_n(P_\bullet) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, a projective resolution is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

where the P_i are projective.

Lemma. Let $\mathcal{A}$ be an abelian category that has enough projectives. Then every object of $\mathcal{A}$ has a projective resolution.

Proof. Given an object A, choose some projective object P_0 with an epimorphism $P_0 \twoheadrightarrow A$. Let $K_0 \rightarrowtail P_0$ be its kernel, and choose P_1 to be a projective object with an epimorphism $P_1 \twoheadrightarrow K_0$, then continue by induction. □

Lemma. Suppose $P_\bullet, Q_\bullet$ are projective resolutions of objects A, B. Then for any $f : A \to B$, there is a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$ with $H_\bullet(f_\bullet) = f$. Moreover, any two such morphisms $P_\bullet \to Q_\bullet$ are homotopic.

Proof. Consider the diagram

$$\begin{tikzcd} P_2 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] & A \arrow[d, "f"] \\ Q_2 \arrow[r] & L_1 \arrow[r] & Q_1 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & B \end{tikzcd}$$

By projectivity of P_0, we obtain f_0 completing the right-hand square.

$$\begin{tikzcd}
P_2 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[d, "f"] \\
Q_2 \arrow[r] & L_1 \arrow[r] & Q_1 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & B
\end{tikzcd}$$

The morphism $P_1 \to P_0 \to A$ is zero by exactness, so $P_1 \to P_0 \to Q_0 \to B$ is also zero. Thus $P_1 \to Q_0$ factors through the kernel $L_0 \to Q_0$. We then obtain f_1 by projectivity.

$$\begin{tikzcd}
P_2 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"'] \arrow[dr] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[d, "f"] \\
Q_2 \arrow[r] & L_1 \arrow[r] & Q_1 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & B
\end{tikzcd}$$

Continue by induction.

Now suppose we have another morphism of chains $g_\bullet$ with $H_0(g_\bullet) = f$. Then $g_0 - f_0$ factors through $L_0 \to Q_0$ as they have the same composite with $Q_0 \to B$. Thus we obtain

$$\begin{tikzcd}
P_2 \arrow[r] & K_1 \arrow[r] & P_1 \arrow[r] \arrow[d] \arrow[dr] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d] \arrow[dll, "h_0"] & A \arrow[d, "f"] \\
Q_2 \arrow[r] & L_1 \arrow[r] & Q_1 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & B
\end{tikzcd}$$

where $d_1' h_0 = g_0 - f_0$. Then

$$d_1'(g_1 - f_1 - h_0 d_1) = d_1' g_1 - d_1' f_1 - d_1' h_0 d_1 = g_0 d_1 - f_0 d_1 - d_1' h_0 d_1 = 0$$

Hence $g_1 - f_1 - h_0 d_1$ factors through $L_1 \to Q_1$, so we obtain h_1 as follows.

$$\begin{tikzcd}
P_2 \arrow[r] \arrow[dr] & K_1 \arrow[r] & P_1 \arrow[r] \arrow[d] \arrow[dr] \arrow[dll, "h_1"] & K_0 \arrow[r] & P_0 \arrow[r] \arrow[d] \arrow[dll, "h_0"] & A \arrow[d, "f"] \\
Q_2 \arrow[r] & L_1 \arrow[r] & Q_1 \arrow[r] & L_0 \arrow[r] & Q_0 \arrow[r] & B
\end{tikzcd}$$

Then $d_2' h_1 + h_0 d_1 = g_1 - f_1$ as required. Continue similarly by induction to construct all components of the homotopy. $\square$

Thus construction of projective resolution is a functor. Note that in this proof we never made use of projectivity of $Q_\bullet$. In particular, this shows that the construction of projective resolutions is left adjoint to $H_0 : \mathcal{C} \to \mathcal{A}$ where $\mathcal{C} \subseteq \mathrm{H}\mathcal{A}$ is the full subcategory on complexes $C_\bullet$ for which $H_n(C_\bullet) = 0$ for all $n > 0$.

7.9 Derived functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then F extends to a functor $\mathrm{C}F : \mathrm{C}\mathcal{A} \to \mathrm{C}\mathcal{B}$ which respects homotopy. Hence F induces a functor $\mathrm{H}F : \mathrm{H}\mathcal{A} \to \mathrm{H}\mathcal{B}$.

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, and suppose $\mathcal{A}$ has enough projectives. Then the *left derived functor* L^nF of F is the composite

$$\mathcal{A} \xrightarrow{\mathrm{PR}} \mathrm{H}\mathcal{A} \xrightarrow{\mathrm{H}F} \mathrm{H}\mathcal{B} \xrightarrow{H_n} \mathcal{B}$$

for any $n \geq 0$, where PR is the projective resolution functor.

Note that if F is exact, we have $L^0F \cong F$ and $L^nF = 0$ for $n > 0$. More generally, if F is right exact, then it preserves exactness of

$$P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

for any projective resolution $P_\bullet$ of A. In particular, $L^0F \cong F$ in this case.

Lemma. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence in an abelian category $\mathcal{A}$ with enough projectives. Then we can choose projective resolutions $P_\bullet, Q_\bullet, R_\bullet$ of A, B, C and morphisms $f_\bullet, g_\bullet$ extending f, g making the sequence

$$0 \longrightarrow P_\bullet \xrightarrow{f_\bullet} Q_\bullet \xrightarrow{g_\bullet} R_\bullet \longrightarrow 0$$

exact. Moreover, the exactness of this sequence is preserved by arbitrary additive functors.

Proof. We choose $P_\bullet, R_\bullet$ arbitrarily, and take $Q_n = P_n \oplus R_n$; this is projective as the coproduct of projective objects is projective. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r, "e_1"] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"'] & A \arrow[d, "f"] \\
& & & P_0 \oplus R_0 \arrow[d, "\begin{pmatrix}0 & 1\end{pmatrix}"'] & B \arrow[d, "g"] \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] & C
\end{tikzcd}$$

By projectivity of R_0, we obtain $h : R_0 \to B$, and so we define $e_2 = \begin{pmatrix} fe_1 & h \end{pmatrix}$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] \\
& & & P_0 \oplus R_0 \arrow[d] \arrow[r, "e_2"] & B \arrow[d, "g"] \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] \arrow[ur, dashed, "h"] & C
\end{tikzcd}$$

This makes both right-hand squares commute:

$$e_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = fe_1; \quad ge_2 = \begin{pmatrix} gfe & gh \end{pmatrix} = \begin{pmatrix} 0 & e_3 \end{pmatrix}$$

To show e_2 is epic, suppose we have a morphism $k : B \to D$ such that $ke_2 = 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] & \\
& & & P_0 \oplus R_0 \arrow[r, "e_2"] \arrow[d] & B \arrow[r, "k"] \arrow[d, "g"] & D \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] & C &
\end{tikzcd}$$

Then $kfe_1 = 0$, so k factors as ℓg for some ℓ.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] & \\
& & & P_0 \oplus R_0 \arrow[r, "e_2"] \arrow[d] & B \arrow[r, "k"] \arrow[d, "g"] & D \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] & C \arrow[ur, "\ell"'] &
\end{tikzcd}$$

Now $\ell e_3 \begin{pmatrix} 0 & 1 \end{pmatrix} = \ell g e_2 = k e_2 = 0$, so $\ell = 0$ as e_3 and $\begin{pmatrix} 0 & 1 \end{pmatrix}$ are pseudoepimorphisms. Thus $k = 0$. Forming the kernel, we obtain

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] \\
& & L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "e_2"] \arrow[d] & B \arrow[d, "g"] \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] & C
\end{tikzcd}$$

Applying the snake lemma to the diagram

$$\begin{tikzcd}
& & 0 \arrow[d] \\
K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] \\
L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "e_2"] \arrow[d] & B \arrow[d, "g"] \\
M_0 \arrow[r] & R_0 \arrow[r, "e_3"] \arrow[d] & C \\
& 0 &
\end{tikzcd}$$

the left-hand column extends to a short exact sequence.

$$0 \longrightarrow K_0 \longrightarrow L_0 \longrightarrow M_0 \longrightarrow 0$$

Hence, as before, we can define an epimorphism $P_1 \oplus R_1 \to L_0$ making the two left-hand squares commute.

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"'] & K_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "e_1"] \arrow[d] & A \arrow[d, "f"] \\
& P_1 \oplus R_1 \arrow[r] \arrow[d, "\begin{pmatrix}0 & 1\end{pmatrix}"'] & L_0 \arrow[r] \arrow[d] & P_0 \oplus R_0 \arrow[r, "e_2"] \arrow[d] & B \arrow[d, "g"] \\
\cdots \arrow[r] & R_1 \arrow[r] & M_0 \arrow[r] & R_0 \arrow[r, "e_3"] & C
\end{tikzcd}$$

Continue by induction. As the columns

$$0 \longrightarrow P_n \longrightarrow Q_n \longrightarrow R_n \longrightarrow 0$$

are biproduct diagrams, they are preserved by arbitrary additive functors. □

This proof does not show that $Q_\bullet \cong P_\bullet \oplus R_\bullet$ in $\mathsf{C}\mathcal{A}$. Indeed, if it were, then $d'_n : Q_n \to Q_{n-1}$ would have matrix

$$\begin{pmatrix} d_n & 0 \\ 0 & d''_n \end{pmatrix}$$

where $d_n : P_n \to P_{n-1}$ and $d''_n : R_n \to R_{n-1}$. Our construction above was of the form

$$\begin{pmatrix} d_n & x \\ 0 & d''_n \end{pmatrix}$$

Theorem. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, and suppose $\mathcal{A}$ has enough projectives. Then, for any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$, we obtain an exact sequence

$$\cdots \longrightarrow L^1FA \longrightarrow L^1FB \longrightarrow L^1FC \longrightarrow L^0FA \longrightarrow L^0FB \longrightarrow L^0FC \longrightarrow 0$$

Proof. Choose projective resolutions $P_\bullet, Q_\bullet, R_\bullet$ for A, B, C as above. Then applying F, we obtain an exact sequence of complexes

$$0 \longrightarrow FP_\bullet \longrightarrow FQ_\bullet \longrightarrow FR_\bullet \longrightarrow 0$$

in $\mathcal{B}$. Then the result follows from the Mayer–Vietoris sequence. □

In particular, L^0F is always right exact, so $L^0F \cong F$ if and only if F is right exact.