Commutative Algebra

Cambridge University Mathematical Tripos: Part III

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Contents

1 Chain conditions

1.1 Modules

In this course, a *ring* is taken to mean a commutative unital ring R. We do however allow for one noncommutative exception, the endomorphism ring $\mathrm{End}(M)$ of an abelian group M. This is a ring where composition is the multiplication operation.

Definition. An R-module is an abelian group M with a fixed ring homomorphism $\rho : R \to \mathrm{End}(M)$. If $r \in R$ and $m \in M$, we define $r \cdot m = \rho(r)(m)$.

Remark. Note that as $\rho(r)$ is a group homomorphism,

$$r(m_1 + m_2) = \rho(r)(m_1 + m_2) = \rho(r)(m_1) + \rho(r)(m_2) = r \cdot m_1 + r \cdot m_2$$

Also, as ρ is a ring homomorphism,

$$(r_1 + r_2)m = \rho(r_1 + r_2)(m) = (\rho(r_1) + \rho(r_2))m = r_1 \cdot m + r_2 \cdot m$$

Example. (i) Let k be a field. Then a k-module is a k-vector space.

(ii) Every abelian group M is a $\mathbb{Z}$-module in a unique way, because the morphism $\mathbb{Z} \to \mathrm{End}\, M$ must map 1 to id.

(iii) Every ring R is an R-module, by taking $\rho(r) = r_0 \mapsto r_0 r$.

Definition. The *direct product* of abelian groups $(M_i)_{i \in I}$ is the set of I-tuples $(a_i)_{i \in I}$ where $a_i \in M_i$, with elementwise addition as the group operation.

Definition. The *direct sum* of abelian groups $(M_i)_{i \in I}$ is the set of I-tuples $(a_i)_{i \in I}$ where $a_i \in M_i$ and all but finitely many of the a_i are zero, again with elementwise addition as the group operation.

Direct products are written $\prod_{i \in I} M_i$, and direct sums are written $\bigoplus_{i \in I} M_i$. These constructions coincide if the index set I is finite. Direct products and direct sums of R-modules are also R-modules.

The universal property of the direct sum states that each collection of module homomorphisms $\varphi_i : M_i \to R$ can be combined into a unique homomorphism $\varphi : \bigoplus_{i \in I} M_i \to R$. Similarly, the universal property of the direct product states that each collection of module homomorphisms $\varphi_i : R \to M_i$ can be combined into a unique homomorphism $\varphi : R \to \prod_{i \in I} M_i$.

1.2 Noetherian and Artinian modules

Definition. An R-module M is *Noetherian* if one of the following conditions holds.

(i) Every ascending chain of submodules $M_0 \subseteq M_1 \subseteq \cdots$ inside M stabilises. That is, for some k, every $j \in \mathbb{N}$ has $M_{k+j} = M_k$.

(ii) Every nonempty set Σ of submodules of M has a maximal element.

Lemma. The two conditions above are equivalent.

Proof. (i) implies (ii). Let Σ be a nonempty set of submodules of M. If it has no maximal element, then for each $M' \in \Sigma$ there exists $M'' \in \Sigma$ with $M' \subsetneq M''$. We can then use the axiom of choice to pick a sequence $M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$ of elements in Σ. This contradicts (i).

(ii) implies (i). Let $M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain of submodules in M. Then let $\Sigma = \{M_0, M_1, \dots\}$. This has a maximal element M_k by (ii). Then for all $j \in \mathbb{N}$, $M_{k+j} = M_k$ as required. $\square$

Definition. M is *Artinian* if one of the following conditions holds.
(i) Every descending chain of submodules $M_0 \supseteq M_1 \supseteq \cdots$ inside M stabilises.
(ii) Every nonempty set Σ of submodules of M has a minimal element.

Again, both conditions are equivalent.

Lemma. An R-module M is Noetherian if and only if every submodule of M is finitely generated.

Proof. Suppose M is Noetherian, and let $N \subseteq M$ be a submodule. Pick $m_1 \in N$, and consider the submodule $M_1 \subseteq N$ generated by m_1. If $M_1 = N$, then we are done. Otherwise, pick $m_2 \in M_1 \setminus N$, and consider $M_2 \subseteq N$ generated by m_2. This construction will always terminate, as if it did not, we would have constructed an infinite strictly ascending chain of submodules of M, contradicting that M is Noetherian.

Now suppose every submodule of M is finitely generated, and let $M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain of submodules of M. Let $N = \bigcup_{i=0}^{\infty} M_i$; this is a submodule of M as the M_i form a chain. Then N is finitely generated, say, by generators $m_1, \dots, m_k \in N$. As the M_i form a chain increasing to N, there exists n such that $m_1, \dots, m_k \in M_n$. In particular, $N \subseteq M_n \subseteq N$, so $M_n = N$. Thus the chain stabilises. $\square$

Note that every Noetherian module is finitely generated. Let $R = \mathbb{Z}[T_1, T_2, \dots]$, and let $M = R$ as an R-module. M is generated by 1_R, so in particular it is finitely generated. But it has a submodule $\langle T_1, T_2, \dots \rangle$ that is not finitely generated. So in the above lemma we indeed must check every submodule.

Definition. A ring R is Noetherian (respectively Artinian) if R is Noetherian (resp. Artinian) as an R-module.

Example. (i) $\mathbb{Z}$ over itself is a Noetherian module as it is a principal ideal domain, but it is not an Artinian module because we can take the chain $(2) \supsetneq (4) \supsetneq (8) \supsetneq \cdots$.

(ii) $\mathbb{Z}$ is similarly a Noetherian ring but not an Artinian ring by unfolding the definition and using (i).

(iii) $\mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$ is an Artinian $\mathbb{Z}$-module but not a Noetherian $\mathbb{Z}$-module. This can be seen from the fact that the only submodules are of the form $\left(\frac{1}{2^k} + \mathbb{Z}\right)$ for $k \in \mathbb{N}$.

(iv) In fact, a ring R is Artinian if and only if R is Noetherian and R has *Krull dimension* 0.

1.3 Exact sequences

Definition. A sequence

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is *exact* if the image of f_i is equal to the kernel of f_{i+1} for each i, where the M_i are modules and the f_i are module homorphisms.

Definition. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{\text{injective}} M \xrightarrow{\text{surjective}} M'' \longrightarrow 0$$

In this situation, $M'' \simeq {}^{M}\!/_{i(M')}$. This is a way to encode M'' as a quotient by a submodule.

Lemma. Let

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\varphi} L \longrightarrow 0$$

be a short exact sequence of R-modules. Then M is Noetherian (resp. Artinian) if and only if both N and L are Noetherian (resp. Artinian).

Proof. We show the statement for Noetherian modules.

Suppose M is Noetherian. If $N_0 \subseteq N_1 \subseteq \cdots$ is an ascending chain of submodules inside N, then by taking images,

$$\iota(N_0) \subseteq \iota(N_1) \subseteq \cdots$$

is also naturally an ascending chain of submodules inside M, so it stabilises. As ι is injective, the original sequence also stabilises. Hence N is Noetherian.

If $L_0 \subseteq L_1 \subseteq \cdots$ is an ascending chain of submodules inside L, then by taking preimages,

$$\varphi^{-1}(L_0) \subseteq \varphi^{-1}(L_1) \subseteq \cdots$$

is an ascending chain of submodules inside M, where

$$\varphi^{-1}(L_i) = \{m \in M \mid \varphi(m) \in L_i\}$$

So this chain stabilises at $\varphi^{-1}(L_k)$. But as φ is surjective, $\varphi(\varphi^{-1}(L_i)) = L_i$, so the original sequence must stabilise at L_k.

Now suppose N and L are Noetherian, and let $M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain of submodules in M. Then

$$\iota^{-1}(M_0) \subseteq \iota^{-1}(M_1) \subseteq \cdots$$

is an ascending chain of submodules in N, so stabilises at $\iota^{-1}(M_{k_N})$ for some k_N. Similarly,

$$\varphi(M_0) \subseteq \varphi(M_1) \subseteq \cdots$$

is an ascending chain of submodules in L, so stabilises at $\varphi{-}1(M_{k_L})$ for some k_L. Take $k \geq k_N, k_L$, and let $j \geq 0$. We show $M_{k+j} \subseteq M_k$, proving that the sequence stabilises.

Let $m \in M_{k+j}$. As $\varphi(M_{k+j}) = \varphi(M_k)$, there exists $m' \in M_k$ such that $\varphi(m) = \varphi(m')$. Then $\varphi(m - m') = 0$, so by exactness, $m - m'$ is in the image of ι, say, $\iota(x) = m - m'$. Since $m - m' \in M_{k+j}$, we must have $x \in \iota^{-1}(M_{k+j})$. But then $x \in \iota^{-1}(M_k)$, so $\iota(x) = m - m' \in M_k$. Hence $m \in M_k$. □

Corollary. If $M_1, \dots, M_n$ are Noetherian (resp. Artinian) modules, then so is $M_1 \oplus \cdots \oplus M_n$.

Proof. Consider the sequence

$$0 \longrightarrow M_1 \xrightarrow{\iota} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0$$

where $\iota(x) = (x, 0)$ and $\pi(x, y) = y$. This is exact, so $M_1 \oplus M_2$ is Noetherian. We then proceed by induction on n. □

Proposition. For a Noetherian (resp. Artinian) ring R, every finitely generated R-module is Noetherian (resp. Artinian).

Proof. M is finitely generated if and only if there is a surjective module homomorphism $\varphi : R^n \to M$ for some $n \geq 0$. That is, M is a quotient of R^n. The fact that R^n is Noetherian (or Artinian) passes through to its quotients. □

1.4 Algebras

Definition. An *R-algebra* is a ring A together with a fixed ring homomorphism $\rho : R \to A$.

Example. The map $k \to k[T_1, \dots, T_n]$ makes the polynomial ring $k[T_1, \dots, T_n]$ a k-algebra.

We will write $ra = \rho(r)a$. Note that $\rho(r) = \rho(r) \cdot 1_A = r \cdot 1_A$, so we can write $r \cdot 1_A$ for $\rho(r)$.

Remark. Every R-algebra is an R-module.

Example. As a k-module, $k[T_1, \dots, T_n]$ is infinite-dimensional. As a k-algebra, $k[T_1, \dots, T_n]$ is generated by the n elements $T_1, \dots, T_n$.

Definition. $\varphi : A \to B$ is an *R-algebra homomorphism* if φ is a ring homomorphism and preserves all elements of R. That is, $\varphi(r \cdot 1_A) = r \cdot 1_B$.

An R-algebra A is finitely generated if and only if there is some $n \geq 0$ and a surjective algebra homomorphism $R[T_1, \dots, T_n] \to A$.

Theorem (Hilbert's basis theorem)**.** Every finitely generated algebra A over a Noetherian ring R is Noetherian.

For example, the polynomial algebra over a field is Noetherian.

Proof. It suffices to prove this for a polynomial ring, as every finitely generated algebra is a quotient of a polynomial ring. It further suffices to prove this for a univariate polynomial ring $A = R[T]$ by induction. Let $\mathfrak{a}$ be an ideal of $R[T]$; we need to show that $\mathfrak{a}$ is finitely generated. For each $i \geq 0$, define

$$\mathfrak{a}(i) = \left\{c_0 \mid c_0 T^i + \dots + c_i T^0 \in \mathfrak{a}\right\}$$

Thus $\mathfrak{a}(i)$ is the set of leading coefficients of polynomials of degree i that lie in $\mathfrak{a}$. Each $\mathfrak{a}(i)$ is an ideal in R, and $\mathfrak{a}(i) \subseteq \mathfrak{a}(i+1)$ by multiplying by T. As R is Noetherian, each $\mathfrak{a}(i)$ is a finitely generated ideal, and this ascending chain stabilises at $\mathfrak{a}(m)$, say. Let

$$\mathfrak{a}(i) = (b_{i,1}, \dots, b_{i,n_i})$$

We can choose $f_{i,j}$ of degree i with leading coefficient $b_{i,j}$. Define the ideal

$$\mathfrak{b} = (f_{i,j})_{i \leq m, j \leq n_i}$$

Note that $\mathfrak{b}$ is finitely generated. Defining $\mathfrak{b}(i)$ in the same way as $\mathfrak{a}(i)$, we have

$$\forall i,\ \mathfrak{a}(i) = \mathfrak{b}(i)$$

By construction, $\mathfrak{b} \subseteq \mathfrak{a}$; we claim that the reverse inclusion holds, then the proof will be complete. Suppose that $\mathfrak{a} \not\subseteq \mathfrak{b}$, and take $f \in \mathfrak{a} \setminus \mathfrak{b}$ of minimal degree i. As $\mathfrak{a}(i) = \mathfrak{b}(i)$, there is a polynomial g in $\mathfrak{b}$ of degree i that has the same leading coefficient. Then $f - g$ has degree less than i, and lies in $\mathfrak{a}$. But then by minimality, $f - g \in \mathfrak{b}$, giving $f \in \mathfrak{b}$. $\square$

Therefore, if $S \subseteq {}^{R[T_1, \dots, T_n]}\!/_{I}$ where R is Noetherian, then $(S) = (S_0)$ where $S_0 \subseteq S$ is finite.

2 Tensor products

2.1 Introduction

Let M and N be R-modules. Informally, the tensor product of M and N over R is the set $M \otimes_R N$ of all sums

$$\sum_{i=1}^{\ell} m_i \otimes n_i; \quad m_i \in M, n_i \in N$$

subject to the relations

$$\begin{aligned}
(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n \\
m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2 \\
(rm) \otimes n &= r(m \otimes n) \\
m \otimes (rn) &= r(m \otimes n)
\end{aligned}$$

This is a module that abstracts the notion of bilinearity between two modules.

Example. Consider ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{3\mathbb{Z}}$. In this $\mathbb{Z}$-module,

$$x \otimes y = (3x) \otimes y = x \otimes (3y) = x \otimes 0 = x \otimes (0 \cdot 0) = 0(x \otimes 0) = 0$$

Hence ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}\!/_{3\mathbb{Z}} = 0$.

Example. Now consider $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell$. We will show later that this is isomorphic to $\mathbb{R}^{n+\ell}$.

2.2 Definition and universal property

Definition. A map of R-modules $f : M \times N \to L$ is *R-bilinear* if for each $m_0 \in M$ and $n_0 \in N$, the maps $n \mapsto f(m_0, n)$ and $m \mapsto f(m, n_0)$ are R-linear (or equivalently, a homomorphism of R-modules).

Definition. Let M, N be R-modules. Let $\mathcal{F} = R^{\oplus(M \times N)}$ be the free R-module with coordinates indexed by $M \times N$. Define $K \subseteq \mathcal{F}$ to be the submodule generated by the following set of relations:

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n) \\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2) \\
&r(m, n) - (rm, n) \\
&r(m, n) - (m, rn)
\end{aligned}$$

The *tensor product* $M \otimes_R N$ is ${}^{\mathcal{F}}\!/_{K}$. We further define the R-bilinear map

$$i_{M \otimes N} : M \times N \to M \otimes N; \quad i_{M \otimes N}(m, n) = e_{(m,n)} = m \otimes n$$

Proposition (universal property of the tensor product)**.** The pair $(M \otimes_R N, i_{M \otimes_R N})$ satisfies the following universal property. For every R-module L and every R-bilinear map $f : M \times N \to L$, there exists a unique homomorphism $h : M \otimes_R N \to L$ such that the following diagram commutes.

$$\begin{tikzcd}
M \times N \arrow[r, "i_{M \otimes_R N}"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, dashed, "h"] \\
 & L
\end{tikzcd}$$

Equivalently, $h \circ i_{M \otimes_R N} = f$.

Proof. The conclusion $h \circ i_{M \otimes N} = f$ holds if and only if for all m, n, we have

$$h(m \otimes n) = f(m, n)$$

Note that the elements $\{m \otimes n\}$ generate $M \otimes N$ as an R-module, so there is at most one h. We now show that the definition of h on the pure tensors $m \otimes n$ extends to an R-linear map $M \otimes N \to L$. The map $R^{\oplus(M \times N)} \to L$ given by $(m, n) \mapsto f(m, n)$ exists by the universal property of the direct sum. However, this map vanishes on the generators of K, so it factors through the quotient ${}^{\mathcal{F}}\!/_{K}$ as required. $\square$

The universal property given above characterises the tensor product up to isomorphism.

Proposition. Let M, N be R-modules, and (T, j) be an R-module and an R-bilinear map $M \times N \to T$. Suppose that (T, j) satisfies the same universal property as $M \otimes N$. Then there is a unique isomorphism of R-modules $\varphi : M \otimes N \xrightarrow{\sim} T$ such that $\varphi \circ i_{M \otimes N} = j$.

Proof. By using the universal property of $M \otimes N$ and T, we obtain φ and ψ as follows.

$$\begin{tikzcd} M \otimes N \arrow[rr, dashed, shift left, "\varphi"] & & T \arrow[ll, dashed, shift left, "\psi"] \\ & M \times N \arrow[ul, "i_{M\otimes N}"] \arrow[ur, "j"'] & \end{tikzcd}$$

The universal property states that $\varphi \circ i_{M\otimes N} = j$ and $\psi \circ j = i_{M\otimes N}$. Hence, $\psi \circ \varphi \circ i_{M\otimes N} = i_{M\otimes N}$. This means that the following diagram commutes.

$$\begin{tikzcd} M \times N \arrow[r, "i_{M\otimes N}"] \arrow[dr, "i_{M\otimes N}"'] & M \otimes N \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\psi\circ\varphi"] \\ & M \otimes N \end{tikzcd}$$

By the uniqueness condition of the universal property, $\mathrm{id} = \psi \circ \varphi$. Similarly, $\mathrm{id} = \varphi \circ \psi$. Hence, φ is an isomorphism $M \otimes N \to T$ with $\varphi \circ i_{M\otimes N} = j$. Uniqueness of φ is guaranteed by the universal property: it is the only solution to $\varphi \circ i_{M\otimes N} = j$. □

In particular, we have

$$\mathrm{Bilin}_R(M \times N, L) \xrightarrow{\sim} \mathrm{Hom}(M \otimes_R N, L)$$

given by the universal property, and the inverse is given by $h \mapsto h \circ i_{M\otimes N}$.

2.3 Zero tensors

Proposition. Let M, N be R-modules. Then

$$\sum m_i \otimes n_i = 0$$

if and only if for every R-module L and every R-bilinear map $f : M \times N \to L$, we have

$$\sum f(m_i, n_i) = 0$$

To show an element of $M \otimes N$ is nonzero, it suffices to find a single R-module L and bilinear map $M \times N \to L$ with mapping the required sum to a nonzero value.

Proof. Assume $\sum m_i \otimes n_i = 0$. f factors through the map $i_{M\otimes N}$, giving

$$\begin{tikzcd} M \times N \arrow[r, "i_{M\otimes N}"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "h"] \\ & L \end{tikzcd}$$

So

$$\sum f(m_i, n_i) = \sum h(i_{M\otimes N}(m_i, n_i)) = h\Big(\sum i_{M\otimes N}(m_i, n_i)\Big) = h(0) = 0$$

In the other direction, suppose $\sum m_i \otimes n_i \neq 0$. Then, taking $f = i_{M\otimes N}$, we obtain $\sum i_{M\otimes N}(m_i, n_i) \neq 0$ as required. □

Example. Let k be a field, and consider $k^m \otimes k^\ell$. Let k^m have basis $\{e_1, \dots, e_m\}$ and k^ℓ have basis $f_1, \dots, f_\ell$. Then

$$k^m \otimes k^\ell = \operatorname{span}_k \left\{ v \otimes w \mid v \in k^m, w \in k^\ell \right\} = \operatorname{span}_k \left\{ e_i \otimes f_j \right\}$$

This is in fact a basis. Suppose $\sum_{i,j} \alpha_{i,j} e_i \otimes f_j = 0$. For each $a \leq m, b \leq \ell$, define $T_{a,b} : k^m \times k^\ell \to k$ by

$$T_{a,b}\left((v_i)_{i=1}^k, (w_j)_{j=1}^\ell\right) = v_a w_b$$

By the above proposition,

$$0 = \sum_{i,j} \alpha_{i,j} T_{a,b}(e_i, f_j) = \alpha_{a,b}$$

So $k^m \otimes k^\ell \simeq k^{m\ell}$. Note that this construction only relied on the existence of a free basis, not on k being a field.

Example. Consider $\mathbb{R}^2 \otimes_\mathbb{R} \mathbb{R}^2$. There are infinitely many pure tensors, but there is a basis consisting of the four pure vectors

$$e_1 \otimes f_1; \quad e_1 \otimes f_2; \quad e_2 \otimes f_1; \quad e_2 \otimes f_2$$

A pure tensor in $\mathbb{R}^2 \otimes_\mathbb{R} \mathbb{R}^2$ is of the form

$$(\alpha e_1 + \beta e_2) \otimes (\gamma f_1 + \delta f_2)$$

which expands to

$$(\alpha\gamma)(e_1 \otimes f_1) + (\alpha\delta)(e_1 \otimes f_2) + (\beta\gamma)(e_2 \otimes f_1) + (\beta\delta)(e_2 \otimes f_2)$$

Note that there is a linear dependence relation between the coefficients $\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta$, so in some sense 'most' tensors are not pure. For example,

$$1(e_1 \otimes f_1) + 2(e_1 \otimes f_2) + 3(e_2 \otimes f_1) + 4(e_2 \otimes f_2)$$

is not pure.

Example. Consider $\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/2\mathbb{Z}$. In this module,

$$2 \otimes (1 + 2\mathbb{Z}) = 1 \otimes (2 + 2\mathbb{Z}) = 1 \otimes 0 = 0$$

Note that $\mathbb{Z}$ has a $\mathbb{Z}$-submodule $2\mathbb{Z}$. In $2\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/2\mathbb{Z}$, the element also denoted with $2 \otimes (1 + 2\mathbb{Z})$ is nonzero. For example, we can define a bilinear map to $\mathbb{Z}/2\mathbb{Z}$ given by

$$b(2n, x + 2\mathbb{Z}) = nx + 2\mathbb{Z}$$

Then $b(2, 1 + 2\mathbb{Z}) = 1 \neq 0$. So it is not the case that tensor products of submodules are submodules of tensor products.

However, if $M' \subseteq M$ and $N' \subseteq N$ and $\sum m_i \otimes n_i = 0$ in $M' \otimes N'$, then $\sum m_i \otimes n_i = 0$ in $M \otimes N$.

Proposition. If $\sum m_i \otimes n_i = 0$ in $M \otimes_R N$, then there are finitely generated R-submodules $M' \subseteq M$ and $N' \subseteq N$ such that the expression $\sum m_i \otimes n_i$ also evaluates to zero in $M' \otimes_R N'$.

This is the last proof that will use the direct construction of the tensor product instead of the universal property directly.

Proof. We know that $\sum m_i \otimes n_i = 0$ in $M \otimes_R N = {R^{\oplus(M\times N)}}/{K}$, so in particular $\sum e_{(m_i,n_i)} \in K$, where e_x maps $x \in M \times N$ to its basis element in $R^{\oplus(M\times N)}$. So this is a finite sum of $\alpha_i k_i$ with $\alpha_i \in R, k_i \in K$, and so we can take the $m_1', \dots, m_a'$ that appear on the left-hand sides of the k_i as the generators for M', and similarly for N'. $\square$

Corollary. Let A, B be torsion-free abelian groups. Then $A \otimes_{\mathbb{Z}} B$ is torsion-free.

Proof. Suppose $n(\sum a_i \otimes b_i) = 0$ with $n \geq 1$. By the previous proposition, there are finitely generated subgroups $A' \leq A$ and $B' \leq B$ such that $n(\sum a_i \otimes b_i) = 0$ in $A' \otimes_{\mathbb{Z}} B'$. But as A' and B' are finitely generated abelian groups, the structure theorem shows that $A' = \mathbb{Z}^m$ and $B' = \mathbb{Z}^\ell$, showing that $A' \otimes_{\mathbb{Z}} B' \simeq \mathbb{Z}^{m\ell}$ is torsion-free. Thus $\sum a_i \otimes b_i = 0$ in $A' \otimes_{\mathbb{Z}} B'$, so also $\sum a_i \otimes b_i = 0$ in $A \otimes_{\mathbb{Z}} B$. $\square$

Example.

$$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3 \simeq \mathbb{C}^6 \simeq \mathbb{R}^{12}$$

However,

$$\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}^3 \simeq \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^6 \simeq \mathbb{R}^{24}$$

This is to be expected: tensoring over a larger ring introduces more relations, so the amount of distinguishable elements should shrink.

2.4 Monoidal structure

We will prove a number of elementary propositions in detail to show how tensor products are used in practice.

Proposition (commutativity)**.** There is an isomorphism $M \otimes N \simeq N \otimes N$ mapping a pure tensor $m \otimes n$ to $n \otimes m$.

Proof. Define $f : M \times N \to N \otimes M$ by $f(m,n) = n \otimes m$; this is bilinear. The universal property yields

$$\begin{tikzcd}
M \times N \arrow[r, "i_{M\otimes N}"] \arrow[dr, "f"'] & M \otimes N \arrow[d, "h"] \\
& N \otimes M
\end{tikzcd}$$

such that $h(m \otimes n) = n \otimes m$. Similarly, we obtain $h' : N \otimes M \to M \otimes N$ with $h'(n \otimes m) = m \otimes n$. Hence, the following diagram commutes.

$$\begin{tikzcd}
M \times N \arrow[r, "i_{M\otimes N}"] \arrow[dr, "i_{M\otimes N}"'] & M \otimes N \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "h'\circ h"] \\
& M \otimes N
\end{tikzcd}$$

So by the uniqueness condition in the universal property, $h' \circ h$ is the identity. Similarly, $h \circ h'$ is the identity, thus h is an isomorphism. $\square$

Proposition (associativity)**.** There is an isomorphism $(M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$ mapping $(m \otimes n) \otimes p$ to $m \otimes (n \otimes p)$.

Proof. For each $p \in P$, define the bilinear map $f_p : M \times N \to M \otimes (N \otimes P)$ by

$$f_p(m, n) = m \otimes (n \otimes p)$$

Thus, each f_p factors through $h_p : M \otimes N \to M \otimes (N \otimes P)$. Then, define the bilinear map $f : (M \otimes N) \times P \to M \otimes (N \otimes P)$ by

$$f(x, p) = h_p(x)$$

We show this is bilinear in p. Note that

$$\begin{aligned} h_{p_1+p_2}(m \otimes n) &= f_{p_1+p_2}(m, n) \\ &= m \otimes (n \otimes (p_1 + p_2)) \\ &= m \otimes (n \otimes p_1) + m \otimes (n \otimes p_2) \\ &= f_{p_1}(m, n) + f_{p_2}(m, n) \\ &= h_{p_1}(m \otimes n) + h_{p_2}(m \otimes n) \end{aligned}$$

So $h_{p_1+p_2}$ coincides with $h_{p_1} + h_{p_2}$ on the pure tensors, so by the universal property they coincide everywhere. Similarly,

$$\begin{aligned} h_{rp}(m \otimes n) &= f_{rp}(m, n) \\ &= m \otimes (n \otimes rp) \\ &= r(m \otimes (n \otimes p)) \\ &= rf_p(m, n) \\ &= rh_p(m \otimes n) \end{aligned}$$

so $h_{rp} = rh_p$. Then, by the universal property, f factors through $h : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$, so

$$h((m \otimes n) \otimes p) = m \otimes (n \otimes p)$$

We can similarly construct $h' : M \otimes (N \otimes P) \to (M \otimes N) \otimes P$ with

$$h'(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$$

Since $h \circ h'$ and $h' \circ h$ are the identity on pure vectors, they are the identity everywhere, and hence are inverse isomorphisms. $\square$

Proposition (identity)**.** There is an isomorphism $R \otimes M \simeq M$ mapping $r \otimes m$ to rm.

Proof. The map $f : R \times M \to M$ given by $f(r, m) = rm$ factors through some $h : R \otimes M \to M$.

$$\begin{tikzcd} R \times M \arrow[r, "i_{R\otimes M}"] \arrow[dr, "f"'] & R \otimes M \arrow[d, dashed, "h"] \\ & M \end{tikzcd}$$

Now define the R-module homomorphism $h' : M \to R \otimes M$ by $h'(m) = 1 \otimes m = i_{R\otimes M}(1, m)$. Then

$$(h \circ h')(m) = h(i_{R\otimes M}(1, m)) = f(1, m) = m$$

giving $h \circ h' = \mathrm{id}$. Further,

$$(h' \circ h)(r \otimes m) = 1 \otimes h(r \otimes m) = 1 \otimes f(r, m) = 1 \otimes rm = r \otimes m$$

So by the uniqueness condition in the universal property, $h' \circ h$ is the identity, and hence h is an isomorphism. $\square$

These operations, together with coherence conditions, make the category of R-modules into a *braided monoidal category*, where the monoid operation is $\otimes$ and the unit is R.

Proposition (distributivity). There is an isomorphism $\left(\bigoplus_i M_i\right) \otimes P \simeq \bigoplus_i (M_i \otimes P)$ mapping $(m_i)_i \otimes p$ to $(m_i \otimes p)_i$.

Proof. Define f by

$$f((m_i)_i, p) = (m_i \otimes p)_i$$

Then there is a unique h such that the following diagram commutes.

$$\begin{tikzcd} \left(\bigoplus_i M_i\right) \times P \arrow[r, "i_{(\bigoplus_i M_i)\otimes P}"] \arrow[dr, "f"'] & \left(\bigoplus_i M_i\right) \otimes P \arrow[d, dashed, "h"] \\ & \bigoplus_i (M_i \otimes P) \end{tikzcd}$$

For each i, define the map $f_i' : M_i \times P \to \left(\bigoplus_i M_i\right) \otimes P$ by

$$f_i'(m_i, p) = m_i \otimes p$$

By the universal property of the tensor product, this factors through a unique h_i'.

$$\begin{tikzcd} M_i \times P \arrow[r, "i_{M_i\otimes P}"] \arrow[dr, "f_i'"'] & M_i \otimes P \arrow[d, dashed, "h_i'"] \\ & \left(\bigoplus_i M_i\right) \otimes P \end{tikzcd}$$

Then, by the universal property of the direct sum, the h_i' can be combined into a single h', so this diagram commutes for each i.

$$\begin{tikzcd} M_i \otimes P \arrow[r] \arrow[dr, "h_i'"'] & \bigoplus_i (M_i \otimes P) \arrow[d, dashed, "h'"] \\ & \left(\bigoplus_i M_i\right) \otimes P \end{tikzcd}$$

It remains to show that h and h' are inverses. To show $h \circ h' = \mathrm{id}_{\bigoplus_i (M_i \otimes P)}$, it suffices by the universal property of the direct sum to show that $(h \circ h')(x) = x$ for all $x \in M_i \otimes P$, for each i. Then, by the universal property of the tensor product, it further suffices to show this result only for pure tensors.

$$\begin{aligned}
(h \circ h')(m_i \otimes p) &= h(h'(m_i \otimes p)) \\
&= h(h_i'(m_i \otimes p)) \\
&= h(f_i'(m_i, p)) \\
&= h(m_i \otimes p) \\
&= f(m_i, p) \\
&= m_i \otimes p
\end{aligned}$$

To show $h' \circ h = \mathrm{id}_{(\bigoplus_i M_i) \otimes P}$, it suffices by the universal property of the tensor product to show that $(h' \circ h)((m_i)_i \otimes p) = (m_i)_i \otimes p$. By linearity of h and h', we can reduce to the case where $(m_i)_i$ has a single non-zero element m_i.

$$\begin{aligned}
(h' \circ h)(m_i \otimes p) &= h'(h(m_i \otimes p)) \\
&= h'(f(m_i, p)) \\
&= h'(m_i \otimes p) \\
&= h_i'(m_i \otimes p) \\
&= f_i'(m_i \otimes p) \\
&= f_i'(m_i, p) \\
&= m_i \otimes p
\end{aligned}$$

□

Example.

$$R^m \otimes_R R^\ell = \left(\bigoplus_{i=1}^m R \right) \otimes_R \left(\bigoplus_{j=1}^\ell R \right) \simeq \bigoplus_{i=1}^m \bigoplus_{j=1}^\ell (R \otimes R) \simeq \bigoplus_{i=1}^m \bigoplus_{j=1}^\ell R \simeq R^{m\ell}$$

Proposition (quotients)**.** Let $M' \subseteq M$ and $N' \subseteq N$ be R-modules. Then there is an isomorphism

$$M/_{M'} \otimes N/_{N'} \simeq (M \otimes N)/_{L}$$

where L is the submodule of $M \otimes N$ generated by

$$\{m' \otimes n \mid (m', n) \in M' \times N\} \cup \{m \otimes n' \mid (m, n') \in M \times N'\}$$

and mapping

$$(m + M') \otimes (n + N') \mapsto m \otimes n + L$$

Proof. Define

$$f : M/_{M'} \times N/_{N'} \to (M \otimes N)/_{L}$$

by

$$f(m + M', n + N') = m \otimes n + L$$

This is well-defined: if $m \in M'$ or $n \in N'$, then $m \otimes n \in L$. By the universal property of the tensor product, f factors through some h.

$$\begin{tikzcd}
M/M' \times N/N' \arrow[r, "i_{M/M' \otimes N/N'}"] \arrow[dr, "f"'] & M/M' \otimes N/N' \arrow[d, dashed, "h"] \\
& (M \otimes N)/L
\end{tikzcd}$$

Now define

$$f' : M \times N \to M/M' \otimes N/N'$$

by

$$f'(m, n) = (m + M') \otimes (n + N')$$

This is clearly bilinear. Thus, we have

$$\begin{tikzcd}
M \times N \arrow[r, "i_{M \otimes N}"] \arrow[dr, "f'"'] & M \otimes N \arrow[d, dashed, "h'"] \\
& M/M' \otimes N/N'
\end{tikzcd}$$

We show that if $x \in L$, then $h'(x) = 0$. By linearity it suffices to show this for the generators.

$$h'(m' \otimes n) = f'(m', n) = 0 \otimes (n + N') = 0; \quad h'(m \otimes n') = f'(m, n') = (m + M') \otimes 0 = 0$$

Thus h' factors through the quotient.

$$\begin{tikzcd}
M \otimes N \arrow[r, "\pi"] \arrow[dr, "h'"'] & (M \otimes N)/L \arrow[d, dashed, "h''"] \\
& M/M' \otimes N/N'
\end{tikzcd}$$

We show h and h'' are inverses. To show $h \circ h'' = \mathrm{id}_{(M \otimes N)/L}$, it suffices by the universal properties of the quotient and the tensor product to consider the images of pure tensors under the quotient map π.

$$\begin{aligned}
(h \circ h'')(m \otimes n + L) &= h(h''(\pi(m \otimes n))) \\
&= h(h'(m \otimes n)) \\
&= h(f'(m, n)) \\
&= h((m + M') \otimes (n + N')) \\
&= f(m + M', n + N') \\
&= m \otimes n + L
\end{aligned}$$

To show $h'' \circ h = \mathrm{id}_{M/M' \otimes N/N'}$, it suffices to show the result for expressions of the form $(m + M') \otimes$

$(n + N')$.

$$\begin{aligned}(h'' \circ h)((m + M') \otimes (n + N')) &= h''(h((m + M') \otimes (n + N'))) \\ &= h''(f(m + M', n + N')) \\ &= h''(m \otimes n + L) \\ &= h'(m \otimes n) \\ &= f'(m + M', n + N') \\ &= (m + M') \otimes (n + N')\end{aligned}$$

□

2.5 Tensor products of maps

Proposition. Let $f : M \to M'$ and $g : N \to N'$ be R-module homomorphisms. There is a unique R-module homomorphism $f \otimes g : M \otimes N \to M' \otimes N'$ such that

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$$

Proof. We apply the universal property to the map $T : M \times N \to M \otimes N'$ given by

$$T(m, n) = f(m) \otimes g(n)$$

which can be checked to be R-bilinear. □

Example. We can show

$$(f \otimes g) \circ (h \otimes i) = (f \circ h) \otimes (g \circ i)$$

For example, if $T : k^a \to k^b$ and $S : k^c \to k^d$,

$$T \otimes S : k^a \otimes_k k^c \to k^b \otimes_k k^d$$

is given by

$$(T \otimes S)(e_i \otimes e_j) = (Te_i) \otimes (Se_j) = \sum_{\ell,t} [T]_{\ell i}[S]_{tj}(f_\ell \otimes f_t)$$

where $[T]$ denotes T in the standard basis. Ordering the basis elements of $k^a \otimes k^c$ as

$$e_1 \otimes e_1, \dots, e_1 \otimes e_c, e_2, \otimes e_1, \dots, e_a \otimes e_c$$

and similarly for $k^b \otimes k^d$,

$$[T \otimes S] = \begin{pmatrix} [T]_{11} \cdot [S] & \cdots & [T]_{1a} \cdot [S] \\ \vdots & \ddots & \vdots \\ [T]_{b1} \cdot [S] & \cdots & [T]_{ba} \cdot [S] \end{pmatrix}$$

This is known as the *Kronecker product* of matrices.

Proposition. Let $f : M \to M', g : N \to N'$ be R-module homomorphisms. Then,

(i) if f, g are isomorphisms, then so is $f \otimes g$;

(ii) if f, g are surjective, then so is $f \otimes g$.

Proof. *Part (i).* $f^{-1} \otimes g^{-1}$ is a two-sided inverse for $f \otimes g$, as

$$(f^{-1} \otimes g^{-1}) \circ (f \otimes g) = (f^{-1} \circ f) \otimes (g^{-1} \otimes g) = \mathrm{id}$$

and similarly for the other side.

Part (ii). The image of $f \otimes g$ contains all pure tensors of $M' \otimes N'$, so it must be surjective. □

The analogous result for injectivity does not hold in the general case. Consider $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by p, and $g : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ given by the identity. Here,

$$(f \otimes g)(a \otimes b) = (pa) \otimes b = a \otimes (pb) = a \otimes 0 = 0$$

So $f \otimes g$ is the zero map, but $\mathbb{Z} \otimes \mathbb{Z}/p\mathbb{Z} \simeq \mathbb{Z}/p\mathbb{Z}$ is not the zero ring.

2.6 Tensor products of algebras

Let B, C be R-algebras. The usual tensor product of modules $B \otimes_R C$ can be made into a ring and then an R-algebra. This allows us to define the tensor product of algebras in a natural way. We want the ring structure to satisfy

$$(b \otimes c)(b' \otimes c') = (bb') \otimes (cc')$$

This extends to a well-defined map on all of $B \otimes C$. Indeed, for a fixed $(b, c) \in B \times C$, there is an R-bilinear map $B \times C \to B \otimes C$ given by

$$(b', c') \mapsto (bb') \otimes (cc')$$

so we can use the universal property to extend this to a map $B \otimes C \to B \otimes C$ that acts on pure tensors in the obvious way. One can show that the ring axioms are satisfied. To define the R-algebra structure, we define the ring homomorphism $R \to B \otimes C$ by

$$r \mapsto (r \cdot 1_B) \otimes 1_C = 1_B \otimes (r \cdot 1_C)$$

Example. There is an isomorphism of R-algebras

$$\varphi : R[X_1, \dots, X_n] \otimes_R R[T_1, \dots, T_r] \xrightarrow{\sim} R[X_1, \dots, X_n, T_1, \dots, T_r]$$

An R-basis for the left-hand side as an R-module is given by elements of the form $a \otimes b$ where a and b are monomials. The right hand side has a basis of elements of the form ab, where $a \in R[X_1, \dots, X_n]$ and $b \in R[T_1, \dots, T_r]$ are monomials as above. Mapping $\varphi(a \otimes b) = ab$, we obtain an R-module isomorphism. To check this is an R-algebra isomorphism, we verify multiplication and its action on scalars.

$$\varphi(r \otimes 1) = r \cdot 1; \quad \varphi(1 \otimes 1)$$

and for monomials p_i, q_i, h_i, g_i,

$$
\begin{aligned}
\varphi\left(\left(\sum_i p_i \otimes q_i\right)\left(\sum_j h_j \otimes g_j\right)\right) &= \sum_{i,j} (p_i h_j)(q_i g_j) \\
&= \sum_{i,j} (p_i q_i)(h_j g_j) \\
&= \sum_{i,j} \varphi(p_i \otimes q_i)\varphi(h_j \otimes g_j) \\
&= \left(\sum_i \varphi(p_i \otimes q_i)\right)\left(\sum_j \varphi(h_j g_j)\right) \\
&= \varphi\left(\sum_i p_i \otimes q_i\right)\varphi\left(\sum_j h_j \otimes g_j\right)
\end{aligned}
$$

More generally,

$$
R[X_1, \dots, X_n]/I \otimes R[T_1, \dots, T_r]/J \simeq R[X_1, \dots, X_n] \otimes R[T_1, \dots, T_r]/L \simeq R[X_1, \dots, X_n, T_1, \dots, T_r]/I^e + J^e
$$

where L is constructed as above when quotients were discussed, and I^e is the extension of I in the larger ring $R[X_1, \dots, X_n, T_1, \dots, T_r]$. For example,

$$
\mathbb{C}[X, Y, Z]/(f, g) \otimes_{\mathbb{C}} \mathbb{C}[W, U]/(h) \simeq \mathbb{C}[X, Y, Z, W, U]/(f, g, h)
$$

Proposition (universal property of tensor product of algebras)**.** Let A, B be R-algebras. For every algebra C and R-algebra homomorphisms $f_1 : A \to C$ and $f_2 : B \to C$, there is a unique R-algebra homomorphism $h : A \otimes_R B \to C$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] \arrow[ddr, bend right, "f_1"'] & & B \arrow[dl, "i_B"'] \arrow[ddl, bend left, "f_2"] \\
& A \otimes B \arrow[d, dashed, "h"] & \\
& C &
\end{tikzcd}
$$

where $i_A(a) = a \otimes 1$ and $i_B(b) = 1 \otimes b$. Furthermore, this characterises the triple $(A \otimes_R B, i_A, i_B)$ uniquely up to unique isomorphism.

Proof. $A \otimes_R B$ is generated as an R-algebra by $\{a \otimes 1 \mid a \in A\} \cup \{1 \otimes b \mid b \in B\}$. This implies the uniqueness of h. For existence, we can define an R-bilinear map $A \times B \to C$ by $(a, b) \mapsto f_1(a) f_2(b)$, then apply the universal property of the tensor product of modules. This produces an R-linear map $h : A \otimes B \to C$. It remains to show that this is a homomorphism of algebras. □

Example.

$$\begin{tikzcd}
R[X_1, \dots, X_n] \arrow[dr] \arrow[ddr, bend right, "f_1"'] & & R[T_1, \dots, T_r] \arrow[dl] \arrow[ddl, bend left, "f_2"] \\
& R[X_1, \dots, X_n, T_1, \dots, T_r] \arrow[d] & \\
& C &
\end{tikzcd}$$

An algebra homomorphism from a polynomial ring is defined uniquely by giving its action on its variables, thus

$$R[X_1, \dots, X_n] \otimes R[T_1, \dots, T_r] \simeq R[X_1, \dots, X_n, T_1, \dots, T_r]$$

as was shown above.

Remark. (i) If $f : A \to A', g : B \to B'$ are R-algebra homomorphisms, then $f \otimes g : A \otimes B \to A' \otimes B'$ is not only an R-module homomorphism but is also an R-algebra homomorphism.

(ii) There are R-algebra homomorphisms

(a) ${}^{R}\!/_{I} \otimes {}^{R}\!/_{J} \simeq {}^{R}\!/_{I+J}$;

(b) $A \otimes B \simeq B \otimes A$;

(c) $A \otimes (B \times C) \simeq (A \otimes B) \times (A \otimes C)$;

(d) $A \otimes B^n \simeq (A \otimes B)^n$;

(e) $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$.

2.7 Restriction and extension of scalars

Let $f : R \to S$ be a ring homomorphism. Let M be an S-module. Then we can *restrict scalars* to make M into an R-module by

$$r \cdot m = f(r) \cdot m$$

The composition $R \to S \to \operatorname{End} M$ is a ring homomorphism, so this makes M into an R-module automatically without needing to check axioms.

Example. Let $f : \mathbb{R} \to \mathbb{C}$ be the inclusion. Then any $\mathbb{C}$-module is an $\mathbb{R}$-module.

Now suppose $f : R \to S$ is a ring homomorphism, M is an S-module, and N is an R-module. We can form the R-module $M \otimes_R N$, as M is an R-module by restriction of scalars. *Extension of scalars* shows that $M \otimes_R N$ is also an S-module. The action of $s \in S$ on pure tensors is

$$s \cdot (m \otimes n) = sm \otimes n$$

We have an R-bilinear map $M \times N \to M \otimes_R N$ by

$$(m, n) \mapsto sm \otimes n$$

so by the universal property this gives rise to a map $h_s : M \otimes_R N \to M \otimes_R N$ with the desired action on pure tensors. h_s is R-linear by the universal property. Defining $\varphi : S \to \operatorname{End}(M \otimes_R N)$ by $\varphi(s) = h_s$, one can check that h_s is a well-defined endomorphism and that φ is a ring homomorphism.

Example. $S \otimes_R R \simeq S$ as R-modules, by $s \otimes r \mapsto s \cdot f(r)$. This is also S-linear, since

$$s'(s \otimes r) = (s's \otimes r) \mapsto s's \cdot f(r) = s'(s \cdot f(r))$$

For example, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} \simeq \mathbb{C}$ as $\mathbb{C}$-modules.

Example. Let M be an S-module and $(N_i)_{i \in I}$ are R-modules. Then

$$M \otimes \left(\bigoplus_i N_i \right) \simeq \bigoplus_i (M \otimes N_i)$$

as S-modules. So $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \simeq \mathbb{C}^n$ as $\mathbb{C}$-modules.

Example. Restrict the $\mathbb{C}$-module $\mathbb{C}^n$ to an $\mathbb{R}$-module to obtain $\mathbb{R}^{2n}$. Then, extending to $\mathbb{C}$,

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{2n} \simeq \mathbb{C}^{2n}$$

Similarly, extending $\mathbb{R}^n$ to $\mathbb{C}$, we find $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \simeq \mathbb{C}^n$ over $\mathbb{C}$. Restricting to $\mathbb{R}$, $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. So the operations of restriction and extension of scalars are not inverses in either direction.

Example. Consider $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. Consider the quotient map $f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Extending scalars to $\mathbb{Z}/2\mathbb{Z}$,

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}^n \simeq \left(\mathbb{Z}/2\mathbb{Z}\right)^n$$

Example. Consider $\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell$ as a $\mathbb{C}$-module. As $\mathbb{R}$-modules,

$$\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \simeq \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{R}^\ell \simeq \mathbb{R}^{2n\ell} \simeq \mathbb{C}^{n\ell}$$

We would like to make this into an isomorphism of $\mathbb{C}$-modules. We will show that in fact

$$\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \simeq \mathbb{C}^n \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^\ell)$$

where

$$v \otimes u \mapsto v \otimes (1 \otimes u)$$

giving

$$\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \simeq \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^\ell \simeq \mathbb{C}^{n\ell}$$

as $\mathbb{C}$-modules. The isomorphism

$$\mathbb{C}^n \otimes_{\mathbb{R}} \mathbb{R}^\ell \simeq \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^\ell$$

maps a pure tensor $v \otimes u$ to $v \otimes u$.

Proposition. Let M be an S-module and N be an R-module. Then

$$M \otimes_R N \simeq M \otimes_S (S \otimes_R N)$$

as S-modules, where

$$m \otimes n \mapsto m \otimes (1 \otimes n); \quad sm \otimes n \leftarrow m \otimes (s \otimes n)$$

Proof. The map $(m, n) \mapsto m \otimes (1 \otimes n)$ is R-bilinear, so the map f mapping $m \otimes n$ to $m \otimes (1 \otimes n)$ is well-defined as a map of R-modules. We show it is S-linear on pure tensors.

$$f(s(m \otimes n)) = f(sm \otimes n) = sm \otimes (1 \otimes n) = s(m \otimes (1 \otimes n)) = sf(m \otimes n)$$

For a fixed $m \in M$, the map $s \otimes n \mapsto sm \otimes n$ is well-defined and S-linear. This collection of maps is S-linear in its parameter m, so we obtain an S-bilinear map $(m, s \otimes n) \mapsto sm \otimes n$. Hence, we obtain a map g mapping $m \otimes (s \otimes n)$ to $sm \otimes n$, as desired. One can easily check that f and g are inverses on pure tensors. $\square$

Proposition. Let M, M' be S-modules and N, N' be R-modules. Then we have S-module isomorphisms

$$\begin{aligned} M \otimes_R N &\simeq N \otimes_R M \\ (M \otimes_R N) \otimes_R N' &\simeq M \otimes_R (N \otimes_R N') \\ (M \otimes_R N) \otimes_S M' &\simeq M \otimes_S (N \otimes_R M') \\ M \otimes_R \left(\bigoplus_i N_i \right) &\simeq \bigoplus_i (M \otimes_R N_i) \end{aligned}$$

Heuristically, the tensor products in the above isomorphisms always operate over the largest possible ring: S if both operands are S-modules, else R. We prove only the third result.

Proof. By the previous proposition,

$$\begin{aligned} (M \otimes_R N) \otimes_S M' &\simeq (M \otimes_S (N \otimes_R S)) \otimes_S M' \\ &\simeq M \otimes_S ((N \otimes_R S) \otimes_S M') \\ &\simeq M \otimes_S (N \otimes_R M') \end{aligned}$$

$\square$

Corollary. Let N, N' be R-modules. Then

$$S \otimes_R (N \otimes_R N') \simeq (S \otimes_R N) \otimes_S (S \otimes_R N')$$

as S-modules.

Proof.
$$S \otimes_R (N \otimes_R N') \simeq (S \otimes_R N) \otimes_R N' \simeq (S \otimes_R N) \otimes_S (S \otimes_R N')$$

$\square$

Example.
$$\mathbb{C} \otimes_{\mathbb{R}} \left(\mathbb{R}^\ell \otimes_{\mathbb{R}} \mathbb{R}^k\right) \simeq (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^\ell) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^k) \simeq \mathbb{C}^\ell \otimes_{\mathbb{C}} \mathbb{C}^k \simeq \mathbb{C}^{\ell k}$$
By induction, one can see that

$$S \otimes_R (N_1 \otimes_R \cdots \otimes_R N_\ell) = (S \otimes_R N_1) \otimes_S \cdots \otimes_S (S \otimes_R N_\ell)$$

2.8 Extension of scalars on morphisms

Let $f : N \to N'$ be an R-linear map, and M be an S-module. Then the map

$$\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$$

is S-linear. Indeed,

$$(\mathrm{id}_M \otimes f)(s(m \otimes n)) = \mathrm{id}_M\, sm \otimes f(n) = s(m \otimes f(n)) = s((\mathrm{id}_M \otimes f)(m \otimes n))$$

Example. Let $T : \mathbb{R}^n \to \mathbb{R}^\ell$ be R-linear, and use bases $e_1, \dots, e_n$ and $f_1, \dots, f_\ell$. Then

$$\mathrm{id}_\mathbb{C} \otimes T : \mathbb{C} \otimes_\mathbb{R} \mathbb{R}^n \to \mathbb{C} \otimes_\mathbb{R} \mathbb{R}^\ell$$

is given by

$$(\mathrm{id}_\mathbb{C} \otimes T)(1 \otimes e_i) = 1 \otimes T(e_i) = 1 \otimes \sum_{j=1}^{\ell} [T]_{ji} \cdot f_j = \sum_{j=1}^{\ell} [T]_{ji} (1 \otimes f_j)$$

This shows that the matrix $[\mathrm{id}_\mathbb{C} \otimes T]$ has all real elements, and is the same as the matrix $[T]$.

2.9 Extension of scalars in algebras

Let A, B be R-algebras. Then the module $A \otimes_R B$ is also an R-algebra. Furthermore, can see that $A \otimes_R B$ is an A-algebra and a B-algebra by the maps $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$.

Example. Consider $R[X_1, \dots, X_n]$ and $f : R \to S$. Then

$$\varphi : S \otimes_R R[X_1, \dots, X_n] \xrightarrow{\sim} S[X_1, \dots, X_n]$$

as S-algebras. Indeed, φ already exists as an isomorphism of S-modules given by

$$\varphi(s \otimes p) = sp$$

and one can verify that unity and multiplication are preserved. Further,

$$S \otimes \left(R[X_1, \dots, X_n] / I \right) \simeq S[X_1, \dots, X_n] / I^e$$

Proposition. Let A be an R-algebra and B be an S-algebra. Then

$$A \otimes_R B \simeq (A \otimes_R S) \otimes_S R$$

as S-algebras.

Proposition. Let A, B be R-algebras. Then

$$S \otimes_R (A \otimes_R B) \simeq (S \otimes_R A) \otimes_S (S \otimes_R B)$$

as S-algebras.

The proofs are omitted, but trivial.

2.10 Exactness properties of the tensor product

Let M be an R-module. There is a functor

$$T_M : \mathbf{Mod}_R \to \mathbf{Mod}_R$$

from the category of R-modules to itself given by

$$T_M(N) = M \otimes_R N; \quad T_M(N \xrightarrow{f} N') = \mathrm{id}_M \otimes f$$

We intend to show that if

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of R-modules, then

$$M \otimes_R A \xrightarrow{T_M(f)} M \otimes_R B \xrightarrow{T_M(g)} M \otimes_R C \longrightarrow 0$$

is also an exact sequence. This shows that T_M is a *right exact* functor.

Definition. Let Q, P be R-modules. Then

$$\mathrm{Hom}_R(Q, P) = \{f : Q \to P \mid f \text{ is } R\text{-linear}\}$$

This is also an R-module: if $\varphi \in \mathrm{Hom}_R(Q, P)$,

$$(r \cdot \varphi)(q) = r \cdot \varphi(q)$$

Definition. Let Q, P be R-modules. Then

$$\mathrm{Hom}_R(Q, -) : \mathbf{Mod}_R \to \mathbf{Mod}_R$$

and

$$\mathrm{Hom}_R(-, P) : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$$

are functors, with action on morphisms $f : N' \to N$ given by

$$\mathrm{Hom}_R(Q, f)(\varphi) = f \circ \varphi = f_\star(\varphi) : \mathrm{Hom}_R(Q, N') \to \mathrm{Hom}_R(Q, N')$$

and

$$\mathrm{Hom}_R(f, P)(\varphi) = \varphi \circ f = f^\star(\varphi) : \mathrm{Hom}_R(N, Q) \to \mathrm{Hom}_R(N', Q)$$

Proposition. Suppose

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Then, so is

$$0 \longrightarrow \mathrm{Hom}_R(Q, A) \xrightarrow{f_\star} \mathrm{Hom}_R(Q, B) \xrightarrow{g_\star} \mathrm{Hom}_R(Q, C)$$

Thus, the covariant hom-functor is *left exact*.

Proof. First, we show $f_\star$ is injective. Suppose $f_\star(\varphi) = 0$, so $f \circ \varphi = 0$. Then as f is injective, $f(\varphi(x)) = 0$ implies $\varphi(x) = 0$, giving $\varphi = 0$ as required.

Now consider $\varphi : Q \to A$. Then

$$g_\star(f_\star(\varphi)) = g \circ (f \circ \varphi) = (g \circ f) \circ \varphi = 0 \circ \varphi = 0$$

so $\operatorname{im} f_\star \subseteq \ker g_\star$. Now suppose $\varphi : Q \to B$ has $g_\star(\varphi) = g \circ \varphi = 0$. So for all $x \in Q$, $g(\varphi(x)) = 0$. By exactness of the original sequence, $\varphi(x) \in \operatorname{im} f$. As f is injective, $\varphi(x)$ has a unique preimage $\psi(x)$ under f. As f is R-linear, so is $\psi : Q \to A$. Hence $f_\star(\psi) = \varphi$ as required. □

Proposition. Suppose

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact. Then, so is

$$0 \longrightarrow \operatorname{Hom}_R(C, P) \xrightarrow{g^\star} \operatorname{Hom}_R(B, P) \xrightarrow{f^\star} \operatorname{Hom}_R(A, P)$$

Thus, the contravariant hom-functor is also left-exact.

Proof. First, we show $g^\star$ is injective. Suppose $g^\star(\varphi) = 0$, so $\varphi \circ g = 0$. As g is surjective, we must have $\varphi = 0$.

Now consider $\varphi : C \to P$. Then

$$f^\star(g^\star(\varphi)) = (\varphi \circ g) \circ f = \varphi \circ (g \circ f) = \varphi \circ 0 = 0$$

so $\operatorname{im} g^\star \subseteq \ker f^\star$. Now suppose $\varphi : B \to P$ has $f^\star(\varphi) = \varphi \circ f = 0$. So for all $x \in A$, $\varphi(f(x)) = 0$. Define $\psi : C \to P$ by

$$\psi(g(x)) = \varphi(x)$$

We show this is well-defined. If $g(x) = g(y)$, then $g(x - y) = 0$, so $x - y = f(a)$ for some $a \in A$. But then $\varphi(f(a)) = 0$, so $\varphi(x) = \varphi(y)$. As φ and g are R-linear, so is ψ. Hence $g^\star(\psi) = \varphi$ as required. □

Lemma. Consider a sequence of R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Suppose that for each R-module P,

$$\operatorname{Hom}_R(C, P) \xrightarrow{g^\star} \operatorname{Hom}_R(B, P) \xrightarrow{f^\star} \operatorname{Hom}_R(A, P)$$

is exact. Then the original sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact.

Proof. First, take $P = C$. By hypothesis, the following sequence is exact.

$$\mathrm{Hom}_R(C, C) \xrightarrow{g^\star} \mathrm{Hom}_R(B, C) \xrightarrow{f^\star} \mathrm{Hom}_R(A, C)$$

Consider

$$\mathrm{id}_C \mapsto \mathrm{id}_C \circ g \mapsto \mathrm{id}_C \circ g \circ f$$

By exactness, id_C must be mapped to zero under $f^\star \circ g^\star$, so $g \circ f = 0$. Hence $\operatorname{im} f \subseteq \ker g$.

Now, take $P = {}^{B}\!/\!_{\operatorname{im} f} = \operatorname{coker} f$.

$$\mathrm{Hom}_R\left(C, {}^{B}\!/\!_{\operatorname{im} f}\right) \xrightarrow{g^\star} \mathrm{Hom}_R\left(B, {}^{B}\!/\!_{\operatorname{im} f}\right) \xrightarrow{f^\star} \mathrm{Hom}_R\left(A, {}^{B}\!/\!_{\operatorname{im} f}\right)$$

Let $h : B \to {}^{B}\!/\!_{\operatorname{im} f}$ be the quotient map. Then,

$$f^\star(h) = h \circ f; \quad h(f(x)) = 0$$

Thus by exactness, h has a preimage $e : C \to {}^{B}\!/\!_{\operatorname{im} f}$. Then $g^\star(e) = e \circ g = h$, so $\ker g \subseteq \ker h = \operatorname{im} f$, giving the reverse inclusion. □

By the universal property of the tensor product,

$$\mathrm{Hom}_R(M \otimes_R N, L) \simeq \mathrm{Bilin}_R(M \times N, L) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_R(M, L))$$

given by

$$\varphi \mapsto (n \mapsto m \mapsto \varphi(m \otimes n)); \quad (m \otimes n \mapsto \varphi(m)(n)) \leftarrow\!\shortmid \varphi$$

This bijection is *natural*, in the sense that many commutative diagrams involving them will commute.

Proposition. Let M be an R-module. Then the functor $T_M = M \otimes_R (-)$ is right exact.

Proof. Consider an exact sequence of R-modules

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

We must show that

$$M \otimes_R A \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_R B \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_R C \longrightarrow 0$$

is exact. Let P be an R-module, and consider apply the functor $\mathrm{Hom}(-, P)$ to this sequence. As this is left exact, the resulting sequence will be exact.

$$0 \longrightarrow \mathrm{Hom}_R(C, P) \xrightarrow{g^\star} \mathrm{Hom}_R(B, P) \xrightarrow{f^\star} \mathrm{Hom}_R(A, P)$$

Then, apply the functor $\mathrm{Hom}(M, -)$, which is also left exact.

$$0 \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_R(C, P)) \xrightarrow{(g^\star)_\star} \mathrm{Hom}_R(M, \mathrm{Hom}_R(B, P)) \xrightarrow{(f^\star)_\star} \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, P))$$

We thus obtain

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(C,P)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(B,P)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_R(M, \mathrm{Hom}_R(A,P)) \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathrm{Hom}_R(M \otimes_R C, P) \arrow[r] & \mathrm{Hom}_R(M \otimes_R B, P) \arrow[r] & \mathrm{Hom}_R(M \otimes_R A, P)
\end{tikzcd}$$

As this diagram commutes, the bottom sequence is exact. Since this holds for all P, by the previous lemma, we can cancel P to give exact sequences

$$0 \longrightarrow M \otimes_R C \longrightarrow M \otimes_R B \qquad M \otimes_R C \longrightarrow M \otimes_R B \longrightarrow M \otimes_R A$$

which combine into the longer sequence as required. $\square$

Remark. It is not the case that if

$$A \longrightarrow B \longrightarrow C$$

is exact, then

$$M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C$$

is also exact; the fact that the sequence has a zero on the right is important. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

and tensor with $\mathbb{Z}/2\mathbb{Z}$. We would then obtain

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\simeq"] & \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\simeq"] & \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z} \arrow[d, "\simeq"] \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

but this sequence is not exact.

2.11 Flat modules

Definition. An R-module M is *flat* if whenever $f : N \to N'$ is R-linear and injective, the map

$$\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$$

is injective.

Example. (i) $\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module.

(ii) Free modules are flat. Suppose $f : N \to N'$ is an injective R-linear map. Then

$$\begin{tikzcd}
R^{\oplus I} \otimes_R N \arrow[r, "\mathrm{id}_{R^{\oplus I}} \otimes f"] \arrow[d, "\simeq"'] & R^{\oplus I} \otimes_R N' \arrow[d, "\simeq"] \\
N^{\oplus I} \arrow[r, "g"'] & (N')^{\oplus I}
\end{tikzcd}$$

commutes, where
$$g((n_i)_{i\in I}) = (f(n_i))_{i\in I}$$
But g is injective, so $\mathrm{id}_{R^{\oplus I}} \otimes f$ must also be injective.

(iii) The base ring matters. One can see that $\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module but it is a flat $\mathbb{Z}/2\mathbb{Z}$-module as it is a free $\mathbb{Z}/2\mathbb{Z}$-module.

Definition. An R-module M is *torsion-free* if $rm \neq 0$ whenever r is not a zero divisor in R and $m \neq 0$.

Proposition. Flat modules are torsion-free.

Proof. Suppose M is not torsion-free. Then there is $r_0 \in R$ not a zero divisor and $m_0 \neq 0$, such that $r_0 m_0 = 0$. Consider the R-linear map $f : R \to R$ given by multiplication by r_0. Its kernel is zero, as r_0 is not a zero divisor. So f is injective. The following diagram commutes.

$$\begin{tikzcd} M \otimes_R R \arrow[r, "\mathrm{id}_M \otimes f"] \arrow[d, "\simeq"'] & M \otimes_R R \arrow[d, "\simeq"] \\ M \arrow[r, "m \mapsto r_0 m"'] & M \end{tikzcd}$$

If M were flat, $\mathrm{id}_M \otimes f$ would be injective, but then the map $m \mapsto r_0 m$ would also be injective, which is a contradiction. $\square$

Example. Let R be an integral domain, and let I be a nonzero ideal of R. Then R/I is not flat. Indeed, if $I = R$ then $R/I = 0$ is not flat. Instead, suppose $I \subsetneq R$, and let $0 \neq x \in I$. Tensoring with R/I, the map $R/I \to R/I$ given by multiplication by x is the zero map, but R/I is not the zero module, so R/I is not torsion-free.

Proposition. Let M be an R-module. Then the following are equivalent.
(i) T_M preserves exactness of all exact sequences;
(ii) T_M preserves exactness of short exact sequences;
(iii) M is flat;
(iv) if $f : N \to N'$ is R-linear and injective, and N, N' are finitely generated R-modules, then $\mathrm{id}_M \otimes f$ is injective.

Note that a map $f : M \to N$ is injective exactly when the sequence
$$0 \longrightarrow M \xrightarrow{f} N$$
is exact, so all of these conditions relate exact sequences.

Proof. Note that (i) implies (ii) which implies (iii) which implies (iv).

(ii) implies (i). Suppose the sequence
$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact. Then, the following diagram is exact.

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \ker f \arrow[dr] & & & & \operatorname{im} g \arrow[ur] \arrow[dr] & & & \\
& & A \arrow[rr, "f"] \arrow[dr] & & B \arrow[rr, "g"] \arrow[ur] & & C \arrow[dr] & & \\
& & & \operatorname{im} f = \ker g \arrow[ur] \arrow[dr] & & & & \operatorname{coker} g \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

After applying $T = T_M$, the diagram still commutes, and the diagonal lines remain exact.

$$\begin{aligned}
\operatorname{im}(TA \to TB) &= \operatorname{im}(TA \to T(\operatorname{im} f) \to TB) \\
&= \operatorname{im}(T(\operatorname{im} f) \to TB) \\
&= \ker(TB \to T(\operatorname{im} g)) \\
&= \ker(TB \to T(\operatorname{im} g) \to TC) \\
&= \ker(TB \to TC)
\end{aligned}$$

(iii) implies (ii). Suppose the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact. As T_M is right exact, we obtain the exact sequence

$$M \otimes_R A \xrightarrow{\mathrm{id}_M \otimes f} M \otimes_R B \xrightarrow{\mathrm{id}_M \otimes g} M \otimes_R C \longrightarrow 0$$

It suffices to show that $\mathrm{id}_M \otimes f$ is injective, but this is precisely the hypothesis of (iii).

(iv) implies (iii). Let $f : N \to N'$ be R-linear and injective. Let $\sum m_i \otimes n_i \in M \otimes_R N$ be such that

$$0 = (\mathrm{id}_M \otimes f)\Big(\sum m_i \otimes n_i\Big) \in M \otimes N'$$

Then there are finitely generated submodules L, L' of N, N' such that the n_i are elements of L and

$$0 = (\mathrm{id}_M \otimes f)\Big(\sum m_i \otimes n_i\Big) \in M \otimes L'$$

By (iv), we obtain

$$0 = \sum m_i \otimes n_i \in M \otimes L$$

But L is a submodule of N, so

$$0 = \sum m_i \otimes n_i \in M \otimes N$$

Hence $\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$ is injective. $\square$

Proposition. Let $f : R \to S$ be a ring homomorphism, and let M be a flat R-module. Then $S \otimes_R M$ is a flat S-module.

Proof. Let $g : N \to N'$ be an S-linear injective map. Then

$$\begin{tikzcd} (S \otimes_R M) \otimes_S N \arrow[r, "\mathrm{id}_{S\otimes_R M} \otimes g"] \arrow[d, "\simeq"'] & (S \otimes_R M) \otimes_S N' \arrow[d, "\simeq"] \\ M \otimes_R N \arrow[r, "\mathrm{id}_M \otimes g"'] & M \otimes_R N' \end{tikzcd}$$

commutes. The map $\mathrm{id}_M \otimes g$ is injective as M is flat, so the map $\mathrm{id}_{S\otimes_R M} \otimes g$ is also injective. Thus $S \otimes_R M$ is a flat S-module. $\square$

We now explore some further examples of tensor products.

Example. Consider $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. In this ring,

$$x \otimes y = n \cdot \frac{x}{n} \otimes y = \frac{x}{n} \otimes ny = \frac{x}{n} \otimes 0 = 0$$

So this ring is trivial. To prove this, we used the fact that for all $x \in \mathbb{Q}$ and $n \geq 1$, there is an element $y \in \mathbb{Q}$ such that $ny = x$. We say that $\mathbb{Q}$ is a *divisible group*. We also needed the fact that $\mathbb{Z}/n\mathbb{Z}$ is a *torsion group*: all elements are of finite order. Hence the tensor product of a divisible group with a torsion group is zero. In particular, it follows that

$$\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$$

However, for an R-module $M \neq 0$, if M is finitely generated then $M \otimes_R M \neq 0$.

Example. Let V be a vector space over $\mathbb{Q}$. Then $\mathbb{Q} \otimes_{\mathbb{Q}} V \simeq V$ as $\mathbb{Q}$-modules, given by the map $x \otimes v \mapsto xv$. However, $\mathbb{Q} \otimes_{\mathbb{Z}} V$ is also isomorphic to V, given by the same map. First, note that every tensor in $\mathbb{Q} \otimes_{\mathbb{Z}} V$ is pure.

$$\sum \frac{a_i}{b_i} \otimes v_i = \sum \frac{1}{b_i} \otimes a_i v_i = \sum \frac{1}{b_i} \otimes b_i \frac{a_i}{b_i} v_i = \sum 1 \otimes \frac{a_i}{b_i} v_i = 1 \otimes \sum \frac{a_i}{b_i} v_i$$

Surjectivity of the map is clear as $1 \otimes v \to v$. We check injectivity on pure tensors. If $xv = 0$, then $x = 0$ or $v = 0$, and in any case, $x \otimes v = 0$.

Example. Consider

$$M \otimes_R \left(\bigoplus_{i\in I} N_i \right) \simeq \bigoplus_{i \in I} (M \otimes_R N_i)$$

given by $m \otimes (n_i)_{i\in I} \mapsto (m \otimes n_i)_{i \in I}$. This is not true with the direct product. However, we do have a map

$$M \otimes_R \left(\prod_{i\in I} N_i \right) \to \prod_{i \in I} (M \otimes_R N_i)$$

given by the same formula, but this is in general not an isomorphism. Consider

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n=1}^{\infty} \mathbb{Z}/2^n\mathbb{Z} \to \prod_{n=1}^{\infty} \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2^n\mathbb{Z} \right)$$

The right-hand side is zero, as each factor is a tensor product of a divisible group by a torsion group. However, the left-hand side is nonzero. Let

$$g = (1, 1, 1, \dots) \in \prod_{n=1}^{\infty} {}^{\mathbb{Z}}\!/_{2^n\mathbb{Z}}$$

This is an element of infinite order, so $\langle g \rangle \simeq \mathbb{Z}$ as a subgroup of $\prod_{n=1}^{\infty} {}^{\mathbb{Z}}\!/_{2^n\mathbb{Z}}$. Thus

$$\mathbb{Q} \otimes_{\mathbb{Z}} \langle g \rangle \simeq \mathbb{Q}$$

as $\mathbb{Z}$-modules. But we have an injective inclusion map

$$\langle g \rangle \to \prod_{n=1}^{\infty} {}^{\mathbb{Z}}\!/_{2^n\mathbb{Z}}$$

We will later show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. This justifies the fact that there is an inclusion

$$\mathbb{Q} \otimes_{\mathbb{Z}} \langle g \rangle \rightarrowtail \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n=1}^{\infty} {}^{\mathbb{Z}}\!/_{2^n\mathbb{Z}}$$

showing that in particular the module in question is nonzero.

Example. Consider $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. We will choose to extend scalars on the left, treating the right-hand copy of $\mathbb{C}$ as an $\mathbb{R}$-module isomorphic to $\mathbb{R}^2$. As a module, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$ is isomorphic to $\mathbb{C}^2$. The basis for $\mathbb{C}^2$ is given by $1 \otimes 1, 1 \otimes i$.

As a $\mathbb{C}$-algebra, we again choose to extend scalars on the left, considering the right-hand copy of $\mathbb{C}$ as an $\mathbb{R}$-algebra.

$$\begin{aligned}
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\simeq \mathbb{C} \otimes_{\mathbb{R}} {}^{\mathbb{R}[T]}\!/_{(T^2+1)} \\
&\simeq {}^{\mathbb{C}[T]}\!/_{(T^2+1)} \\
&\simeq {}^{\mathbb{C}[T]}\!/_{(T-i)(T+i)} \\
&\simeq {}^{\mathbb{C}[T]}\!/_{(T-i)} \times {}^{\mathbb{C}[T]}\!/_{(T+i)} \\
&\simeq \mathbb{C} \times \mathbb{C}
\end{aligned}$$

using the Chinese remainder theorem, which will be explored later. The action of this isomorphism on a pure tensor is

$$\begin{aligned}
x \otimes y = (a+bi) \otimes (c+di) &\mapsto (a+bi) \otimes (c + dT + (T^2+1)\mathbb{R}[T]) \\
&\mapsto (a+bi)(c+dT) + (T^2+1)\mathbb{C}[T] \\
&= \underbrace{(ac + bdiT) + (ibc + adT)}_{P} + (T^2+1)\mathbb{C}[T] \\
&\mapsto (P + (T-i)\mathbb{C}[T], P + (T+i)\mathbb{C}[T]) \\
&\mapsto ((ac-bd) + i(bc+ad), (ac+bd) + i(bc-ad)) = (xy, x\overline{y})
\end{aligned}$$

3 Localisation

3.1 Definitions

Definition. A *multiplicative set* or *multiplicatively closed set* $S \subseteq R$ is a subset such that $1 \in S$ and if $a, b \in S$, then $ab \in S$. If $U \subseteq R$ is any set, its *multiplicative closure* S is the set

$$\left\{ \prod_{i=1}^{n} u_i \,\middle|\, n \geq 0, u_i \in U \right\}$$

which is the smallest multiplicatively closed set containing U.

Example. (i) If R is an integral domain, then $S = R \setminus \{0\}$ is multiplicative.

(ii) More generally, if $\mathfrak{p}$ is a prime ideal in R, then $S = R \setminus \mathfrak{p}$ is multiplicative.

(iii) If $x \in R$, then the set $\{x^n \mid n \geq 0\}$ is multiplicative.

Remark. $\mathbb{Q}$ is obtained from $\mathbb{Z}$ by adding inverses for the elements of the multiplicative subset $\mathbb{Z} \setminus \{0\}$. We have a ring homomorphism $\mathbb{Z} \rightarrowtail \mathbb{Q}$. We generalise this construction to arbitrary rings and multiplicative sets. In general, injectivity of the ring homomorphism in question may fail.

Definition. Let $S \subseteq R$ be a multiplicative set, and let M be an R-module. Then the *localisation* of M by S is the set $S^{-1}M = {}^{M \times S}\!/\!_{\sim}$ where $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $u \in S$ such that $u(s_2 m_1 - s_1 m_2) = 0$. We write $\frac{m}{s}$ for the equivalence class corresponding to (m, s). We make $S^{-1}M$ into an R-module by defining

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{m_1 s_2 + m_2 s_1}{s_1 s_2}; \quad r \cdot \frac{m}{s} = \frac{rm}{s}$$

We can make $S^{-1}R$ into a ring by defining

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}$$

Then $S^{-1}M$ is an $S^{-1}R$-module by

$$\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}$$

We have the localisation map $R \to S^{-1}R$ given by $r \mapsto \frac{r}{1}$, which is a ring homomorphism. We also have the localisation map $M \to S^{-1}M$ given by $m \mapsto \frac{m}{1}$, which is a homomorphism of R-modules.

We must show that $\sim$ is an equivalence relation. The only nontrivial thing to prove is transitivity. Let

$$u(s_2 m_1 - s_1 m_2) = 0 = v(s_3 m_2 - s_2 m_3); \quad u, v \in S$$

Then

$$0 = uv(s_2 s_3 m_1 - s_1 s_3 m_2) + uv(s_1 s_3 m_2 - s_1 s_2 m_3) = uvs_2(s_3 m_1 - s_1 m_3); \quad uvs_2 \in S$$

as required. All other operations mentioned are well-defined; the proofs are not enlightening so are omitted.

3.2 Universal property for rings

Proposition. Let $U \subseteq R$, and let $S \subseteq R$ be its multiplicative closure. Let $f : R \to B$ be a ring homomorphism such that $f(u)$ is a unit for all $u \in U$. Then there is a unique ring homomorphism $h : S^{-1}R \to B$ such that the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "\iota_{S^{-1}R}"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "h"] \\ & B \end{tikzcd}$$

where $\iota_{S^{-1}R}(r) = \frac{r}{1}$, so in particular, $f(r) = h\left(\frac{r}{1}\right)$.

Thus

$$\mathrm{Hom}_{\mathrm{Ring}}(S^{-1}R, B) \simeq \left\{\varphi \in \mathrm{Hom}_{\mathrm{Ring}}(R, B) \mid \varphi(U) \subseteq B^\times\right\}$$

mapping

$$f \mapsto \left(r \mapsto \frac{r}{1}\right); \quad \left(\frac{r}{s} \mapsto \frac{\varphi(r)}{\varphi(s)}\right) \mapsfrom \varphi$$

Proof. Let $f : R \to B$ be a ring homomorphism such that $f(u)$ is a unit for all $u \in U$. Then $f(s)$ is a unit for all $s \in S$. We want to construct a ring homomorphism $h : S^{-1}R \to B$ such that $f(r) = h\left(\frac{r}{1}\right)$ for all $r \in R$. Such an h must satisfy the following condition.

$$1 = h(1) = h\left(\frac{1}{s} \cdot \frac{s}{1}\right) = h\left(\frac{1}{s}\right)f(s)$$

Thus $h\left(\frac{1}{s}\right) = f(s)^{-1}$. Hence, we must have

$$h\left(\frac{r}{s}\right) = h\left(\frac{1}{s}\right)h\left(\frac{r}{1}\right) = f(s)^{-1}f(r)$$

It thus suffices to show that this h is well-defined; it is then a ring homomorphism satisfying the correct property. If $\frac{r_1}{s_1} = \frac{r_2}{s_2}$, then there is $t \in S$ such that $ts_2r_1 = ts_1r_2$. Applying f,

$$f(t)f(s_2)f(r_1) = f(t)f(s_1)f(r_2)$$

As $f(t), f(s_1), f(s_2)$ are invertible,

$$\frac{f(r_1)}{f(s_1)} = \frac{f(r_2)}{f(s_2)}$$

so h is well-defined. □

Proposition. Suppose (A, j) has the same universal property of $(S^{-1}R, \iota_{S^{-1}R})$ where $\iota_{S^{-1}R}(r) = \frac{r}{1}$, then there is a unique ring isomorphism $S^{-1}R \to A$ mapping $\frac{r}{s}$ to $j(s)^{-1}j(r)$.

Remark. (i) Let $\frac{r}{s} \in S^{-1}R$. Then $\frac{r}{s} = \frac{0}{1}$ if and only if there exists $u \in S$ such that $ur = 0$.

(ii) In particular, $S^{-1}R = 0$ when $\frac{1}{1} = \frac{0}{1}$, which occurs precisely when $0 \in S$.

(iii) $\ker \iota_{S^{-1}R} = \{r \in R \mid \exists u \in S,\ ur = 0\}$.

(iv) $\iota_{S^{-1}R}$ is injective if and only if S contains no zero divisors.

(v) $\iota_{S^{-1}R}$ is always an epimorphism, but usually not surjective. For example, the map $\iota : \mathbb{Z} \rightarrowtail \mathbb{Q}$ is epic. Indeed, for $f, g : \mathbb{Q} \to A$ are such that $f \circ \iota = g \circ \iota$, then

$$f\left(\frac{p}{q}\right) = \frac{f(\iota(p))}{f(\iota(q))} = \frac{g(\iota(p))}{g(\iota(q))} = g\left(\frac{p}{q}\right)$$

Example. (i) Let $f \in R$ and define $S = \{f^n \mid n \geq 0\}$. Define $R_f = S^{-1}R$. Taking for instance $R = \mathbb{Z}$ and $f = 2$,

$$R_f = \left\{\frac{a}{2^n} \,\middle|\, a \in \mathbb{Z},\ n \geq 0\right\} = \mathbb{Z}\left[\frac{1}{2}\right]$$

producing the ring of dyadic rational numbers. Since we write $\mathbb{Z}/_{n\mathbb{Z}}$ for the finite quotient ring and $\mathbb{Z}_2$ for the 2-adic integers, we must use the notation $\mathbb{Z}\left[\frac{1}{2}\right]$ for this particular construction instead. Thus R_f is the zero ring if and only if f is nilpotent.

(ii) Let $\mathfrak{p} \in \operatorname{Spec} R$, where $\operatorname{Spec} R$ is the set of prime ideals in R. Then $S = R \setminus \mathfrak{p}$ is a multiplicative set. Consider $(R \setminus \mathfrak{p})^{-1}R = R_{\mathfrak{p}}$. For example,

$$\mathbb{Z}_{(3)} = \left\{\frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ 3 \nmid b\right\}$$

3.3 Functoriality

Proposition. Let M be an R-module and $S \subseteq R$ be a multiplicative set. Then there is an isomorphism of $S^{-1}R$-modules

$$S^{-1}R \otimes_R M \to S^{-1}M$$

given by $\frac{r}{s} \otimes m \mapsto \frac{rm}{s}$.

Thus the localisation of any module can be reduced to a tensor product with the localisation of a ring.

Proof. Define the map $S^{-1}R \times M \to S^{-1}M$ mapping $\left(\frac{r}{s}, m\right) \mapsto \frac{rm}{s}$; this is bilinear and thus gives rise to an R-linear map $\varphi : S^{-1}R \otimes M \to S^{-1}M$ with the desired action on pure tensors. One can check that this is in fact $S^{-1}R$-linear. Clearly φ is surjective by $\frac{1}{s} \otimes m \mapsto \frac{m}{s}$. For injectivity, we first show that every tensor

$$\sum_i \frac{r_i}{s_i} \otimes m_i \in S^{-1}R \otimes_R M$$

is pure. We define

$$s = \prod_i s_i; \quad t_j = \prod_{j \neq i} s_j$$

hence

$$\sum_i \frac{r_i}{s_i} \otimes m_i = \sum_i \frac{1}{s_i} \otimes r_i m_i = \sum_i \frac{t_i}{s} \otimes r_i m_i = \sum_i \frac{1}{s} \otimes t_i r_i m_i = \frac{1}{s} \otimes \sum_i t_i r_i m_i$$

as required. Now, it suffices to prove injectivity on pure tensors. If $\varphi\left(\frac{1}{s} \otimes m\right) = \frac{0}{1}$, then there exists $u \in S$ such that

$$u(1m - 0s) = 0 \implies um = 0$$

Thus

$$\frac{1}{s} \otimes m = \frac{u}{us} \otimes m = \frac{1}{us} \otimes um = \frac{1}{us} \otimes 0 = 0$$

as required. □

The map $S^{-1}R \otimes (-)$ acts on modules and on morphisms. The map $S^{-1}(-)$ acts on modules, and can be extended to act on morphisms in the following way. If $f : N \to N'$ is R-linear, we produce the commutative diagram

$$\begin{tikzcd} S^{-1}R \otimes_R N \arrow[r, "\mathrm{id}_{S^{-1}R} \otimes f"] \arrow[d, "\sim"'] & S^{-1}R \otimes_R N' \arrow[d, "\sim"] \\ S^{-1}N \arrow[r, dashed, "S^{-1}(f)"'] & S^{-1}N' \end{tikzcd}$$

with action

$$\begin{tikzcd} \frac{1}{s} \otimes n \arrow[r, mapsto] & \frac{1}{s} \otimes f(n) \arrow[d, mapsto] \\ \frac{n}{s} \arrow[u, mapsto] \arrow[r, dashed, mapsto] & \frac{f(n)}{s} \end{tikzcd}$$

Then the functor $S^{-1}R \otimes_R (-)$ is naturally isomorphic to the functor $S^{-1}(-)$.

Remark. If A is an R-algebra, then we have an $S^{-1}R$-linear isomorphism $S^{-1}R \otimes_R A \xrightarrow{\sim} S^{-1}A$; this is also an isomorphism of $S^{-1}R$-algebras.

Lemma. Let M be an $S^{-1}R$-module. Treating M as an R-module, we can define $S^{-1}M$. Then,

$$S^{-1}M \simeq M$$

as $S^{-1}R$-modules, mapping $\frac{m}{s} \mapsto \frac{1}{s}m$.

Equivalently, $M \simeq S^{-1}R \otimes_R M$ as $S^{-1}R$-modules, mapping $m \mapsto \frac{1}{1} \otimes m$.

Proof. The localisation map $M \to S^{-1}M$ maps $m \mapsto \frac{m}{1}$. This is $S^{-1}R$-linear, and surjective as $\frac{1}{s} \cdot m \mapsto \frac{m}{s}$. To show injectivity, note that $\frac{m}{1} = \frac{0}{1}$ implies there exists $u \in S$ with $um = 0$. Multiplying by $\frac{1}{u}$ as M is an $S^{-1}R$-module we obtain $m = 0$ as required. □

3.4 Universal property for modules

Recall that if U has multiplicative closure S,

$$\mathrm{Hom}_{\mathrm{Ring}}(S^{-1}R, B) \simeq \left\{\varphi \in \mathrm{Hom}_{\mathrm{Ring}}(R, B) \mid \varphi(U) \subseteq B^\times\right\}$$

If M is a fixed R-module and L is an $S^{-1}R$-module, we have

$$\mathrm{Hom}_R(M, L) \simeq \mathrm{Hom}_{S^{-1}R}(S^{-1}M, L)$$

Proposition. Let M be an R-module and L be an $S^{-1}R$-module. Let $f : M \to L$ be R-linear. Then there exists a unique $S^{-1}R$-linear map $h : S^{-1}M \to L$ such that $f = h \circ i_{S^{-1}M}$.

$$\begin{tikzcd} M \arrow[r, "i_{S^{-1}M}"] \arrow[dr, "f"'] & S^{-1}M \arrow[d, dashed, "h"] \\ & L \end{tikzcd}$$

As usual with universal properties, this characterises $S^{-1}M$ uniquely up to unique isomorphism.

Proof. We use the natural isomorphism between $S^{-1}(-)$ and $S^{-1}R \otimes_R (-)$. After applying this, we have a map

$$\iota : M \to S^{-1}R \otimes_R M; \quad m \mapsto \frac{1}{1} \otimes m$$

Let $f : M \to L$ be R-linear, and define

$$h = \mathrm{id}_{S^{-1}R} \otimes f : S^{-1}R \otimes_R M \to S^{-1}R \otimes_R L$$

Note that $S^{-1}R \otimes_R L \simeq L$, so we can consider h as mapping to L, with action

$$h\Big(\frac{r}{s} \otimes m\Big) = \frac{r}{s} f(m)$$

Uniqueness of h follows from the fact that $\{1 \otimes m\}_{m \in M}$ generate $S^{-1}R \otimes_R M$ as an $S^{-1}R$-module. □

3.5 Exactness

Proposition. The functor $S^{-1}(-)$ is exact. More explicitly, if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence of R-modules, then

$$S^{-1}A \xrightarrow{S^{-1}f} S^{-1}B \xrightarrow{S^{-1}g} S^{-1}C$$

is an exact sequence of $S^{-1}R$-modules.

Proof. First,

$$(S^{-1}g) \circ (S^{-1}f) = S^{-1}(g \circ f) = S^{-1}0 = 0$$

so $\operatorname{im} S^{-1}f \subseteq \ker S^{-1}g$. Now suppose $\frac{b}{s} \in \ker S^{-1}g$, so $\frac{g(b)}{s} = \frac{0}{1}$. Hence there exists $u \in S$ such that $ug(b) = 0$. As g is R-linear and $u \in R$, we have $g(ub) = 0$. By exactness, $ub \in \ker g = \operatorname{im} f$. Thus there exists $a \in A$ such that $f(a) = ub$. Hence,

$$\frac{b}{s} = \frac{ub}{us} = \frac{f(a)}{us} = S^{-1}f\Big(\frac{a}{us}\Big)$$

$\square$

In particular, $S^{-1}R$ is a flat R-module, so for example $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

Remark. Suppose $N \subseteq M$ are R-modules, and $\iota : N \to M$ is the inclusion map. Then applying the localisation, the map $S^{-1}\iota : S^{-1}N \to S^{-1}M$ given by $\frac{n}{s} \mapsto \frac{n}{s}$ is still injective. Note that the similar result for tensor products fails.

Proposition. Let M be an R-module and N, P be submodules of M. Then,

(i) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$;
(ii) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$;
(iii) $S^{-1}M \big/ S^{-1}N \xrightarrow{\sim} S^{-1}\left(M \big/ N\right)$ given by $\frac{m}{s} + S^{-1}N \mapsto \frac{m+N}{s}$.

Parts (i) and (ii) rely on a slight abuse of notation, thinking of $S^{-1}N$ as a submodule of $S^{-1}M$. Due to the above remark, this should not cause confusion.

Proof. *Part (i).* Note that

$$\frac{n+p}{s} = \frac{n}{s} + \frac{p}{s} \in S^{-1}N + S^{-1}P$$

and

$$\frac{n}{s_1} + \frac{p}{s_2} = \frac{s_2 n + s_1 p}{s_1 s_2} \in S^{-1}(N + P)$$

Part (ii). The forward inclusion is clear. Conversely, suppose $x \in S^{-1}N \cap S^{-1}P$, so $x = \frac{n}{s_1} = \frac{p}{s_2}$. Hence, there exists $u \in S$ such that $us_2 n = us_1 p = w$. Note $us_2 n \in N$ and $us_1 p \in P$, so $w \in N \cap P$. Now,

$$x = \frac{n}{s_1} = \frac{us_2 n}{us_1 s_2} = \frac{w}{us_1 s_2} \in S^{-1}(N \cap P)$$

Part (iii). Consider the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\pi} M \big/ N \longrightarrow 0$$

Applying the exact functor $S^{-1}(-)$, we obtain the short exact sequence

$$0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}\iota} S^{-1}M \xrightarrow{S^{-1}\pi} S^{-1}\left(M \big/ N\right) \longrightarrow 0$$

Thus

$$(S^{-1}\iota)(S^{-1}N) = S^{-1}N \subseteq S^{-1}M$$

and

$$(S^{-1}\pi)\left(\frac{m}{s}\right) = \frac{m+N}{s}$$

giving the isomorphism as required. $\square$

Proposition. Let M, N be R-modules. Then

$$S^{-1}M \otimes_{S^{-1}R} S^{-1}N \xrightarrow{\sim} S^{-1}(M \otimes_R N)$$

Proof. We have already proven that

$$(S^{-1}R \otimes_R M) \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \simeq S^{-1}R \otimes_R (M \otimes_R N)$$

giving the result as required. □

Example. Let $\mathfrak{p}$ be a prime ideal in R. Then by setting $S = R \setminus \mathfrak{p}$,

$$M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p} \simeq (M \otimes_R N)_\mathfrak{p}$$

3.6 Extension and contraction of ideals

If $f : A \to B$ is a ring homomorphism and $\mathfrak{b}$ is an ideal in B, the preimage $f^{-1}(\mathfrak{b}) = \mathfrak{b}^c$ is an ideal in A, called its *contraction*. If $\mathfrak{a}$ is an ideal in A, we can generate an ideal $(f(\mathfrak{a})) = \mathfrak{a}^e$ in B, called its *extension*. We show on the first example sheet that for any ring homomorphism $f : A \to B$, there is a bijection

$$\{\text{contracted ideals of } A\} \leftrightarrow \{\text{extended ideals of } B\}$$

noting that the contracted ideals are those ideals with $\mathfrak{a} = \mathfrak{a}^{ec}$, and the extended ideals are those ideals with $\mathfrak{b} = \mathfrak{b}^{ce}$, where the bijection maps $\mathfrak{a} \mapsto \mathfrak{a}^e$ and $\mathfrak{b}^c \leftarrow\!\shortmid \mathfrak{b}$.

We now study the special case where $f : R \to S^{-1}R$ is the localisation map of a ring, given by $r \mapsto \frac{r}{1}$. In this case, the extension of an ideal is written $S^{-1}\mathfrak{a} = \mathfrak{a}^e$. We claim that

$$\mathfrak{a}^e = \left\{ \frac{a}{s} \,\middle|\, a \in \mathfrak{a}, s \in S \right\}$$

Indeed, $\mathfrak{a}^e$ is generated by $\left\{ \frac{a}{1} \,\middle|\, a \in \mathfrak{a} \right\}$, so $\mathfrak{a}^e$ must contain $\left\{ \frac{a}{s} \,\middle|\, a \in \mathfrak{a}, s \in S \right\}$, but this is already an ideal. We also claim that

$$\mathfrak{a}^{ec} = \bigcup_{s \in S} (\mathfrak{a} : s); \quad (\mathfrak{a} : s) = \{ r \in R \mid rs \in \mathfrak{a} \}$$

Indeed, for $r \in \bigcup_{s \in S} (\mathfrak{a} : s)$, we have $rs = a$ in R for some $s \in S$ and $a \in \mathfrak{a}$, so $\frac{rs}{1} = \frac{a}{1}$, giving $\frac{r}{1} = \frac{a}{s}$, so $r \in \mathfrak{a}^{ec}$ as required. In the other direction, if $r \in \mathfrak{a}^{ec}$, then $\frac{r}{1} = \frac{a}{s}$ for some $s \in S$ and $a \in \mathfrak{a}$, so there exists $u \in S$ such that $rus = ua \in \mathfrak{a}$, so $r \in (\mathfrak{a} : us)$ as required.

Now, let $\mathfrak{b}$ be an ideal of $S^{-1}R$. Then

$$\mathfrak{b}^c = \left\{ r \in R \,\middle|\, \frac{r}{1} \in \mathfrak{b} \right\}$$

We claim that $\mathfrak{b}^{ce} = \mathfrak{b}$, so all ideals in $S^{-1}R$ are extended. Note that the inclusion $\mathfrak{b}^{ce} \subseteq \mathfrak{b}$ holds for any pair of rings. For the reverse inclusion, consider $\frac{r}{s} \in \mathfrak{b}$, so $\frac{r}{1} \in \mathfrak{b}$. Hence $r \in \mathfrak{b}^c$, so $\frac{r}{1} \in \mathfrak{b}^{ce}$, thus $\frac{r}{s} \in \mathfrak{b}^{ce}$ as $\mathfrak{b}^{ce}$ is an ideal in $S^{-1}R$.

Proposition. Consider the localisation map $R \to S^{-1}R$ given by $r \mapsto \frac{r}{1}$.

(i) Every ideal of $S^{-1}R$ is extended.
(ii) An ideal $\mathfrak{a}$ of R is contracted if and only if the image of S in $R/\mathfrak{a}$ contains no zero divisors.
(iii) $\mathfrak{a}^e = S^{-1}R$ if and only if $\mathfrak{a} \cap S \neq \varnothing$.
(iv) There is a bijection

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\} \leftrightarrow \operatorname{Spec} S^{-1}R$$

given by $\mathfrak{p} \mapsto \mathfrak{p}^e$, $\mathfrak{q}^c \leftarrow\!\shortmid \mathfrak{q}$.

Proof. *Part (i).* Follows from the fact that $\mathfrak{b}^{ce} = \mathfrak{b}$ for all ideals $\mathfrak{b}$ in $S^{-1}R$.

Part (ii). $\mathfrak{a}$ is contracted if and only if $\mathfrak{a}^{ec} \subseteq \mathfrak{a}$, because the reverse inclusion always holds. This happens if and only if

$$\bigcup_{s\in S}(\mathfrak{a} : s) \subseteq \mathfrak{a}$$

which occurs if and only if

$$\forall r \in R,\ (Sr \cap \mathfrak{a} \neq \varnothing \implies r \in \mathfrak{a})$$
$$\forall r \in R,\ (0 + \mathfrak{a} \in S(r + \mathfrak{a}) \implies r + \mathfrak{a} = 0 + \mathfrak{a})$$

which in turn occurs if and only if the image of S in $R/\mathfrak{a}$ contains no zero divisors.

Part (iii). Suppose $\mathfrak{a} \cap S \neq \varnothing$, so let $x \in \mathfrak{a} \cap S$. Then $\frac{x}{x} \in \mathfrak{a}^e$, so $\mathfrak{a}^e = (1) = S^{-1}R$. Conversely, if $\mathfrak{a}^e = S^{-1}R$, then $\frac{1}{1} \in \mathfrak{a}^e$, so $\frac{1}{1} = \frac{a}{s}$ for some $a \in \mathfrak{a}$, $s \in S$. Therefore there exists $u \in S$ such that $us = ua \in S \cap \mathfrak{a}$.

Part (iv). Consider the contraction map $\operatorname{Spec} S^{-1}R \to \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\}$ given by $\mathfrak{q} \mapsto \mathfrak{q}^c$. We show this is well-defined. In general, a contraction of a prime ideal is always prime. Further, $\mathfrak{p} \in \operatorname{Spec} R$ is contracted if and only if the image of S in $R/\mathfrak{p}$ contains no zero divisors, but $R/\mathfrak{p}$ is an integral domain, so its only zero divisor is zero itself. So this condition is equivalent to the condition $\mathfrak{p} \cap S = \varnothing$. In particular, $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\}$ is precisely the set of contracted prime ideals of R. The map is injective, since if $\mathfrak{q} \in \operatorname{Spec} S^{-1}R$, then $\mathfrak{q}^{ce} = \mathfrak{q}$.

In the other direction, for $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} \cap S = \varnothing$, it must be contracted, so $\mathfrak{p}^{ec} = \mathfrak{p}$. It therefore remains to show that $\mathfrak{p}^e$ is a prime ideal. We want to show that $S^{-1}R/\mathfrak{p}^e$ is an integral domain. We have that $\mathfrak{p}^e \neq S^{-1}R$ by (iii), so $S^{-1}R/\mathfrak{p}^e$ is not the zero ring, so it suffices to show that this quotient has no zero divisors. To show this, we embed $S^{-1}R/\mathfrak{p}^e$ in the field $FF(R/\mathfrak{p})$.

Consider the composite map

$$R \to R/\mathfrak{p} \to FF(R/\mathfrak{p})$$

which is a surjection followed by an injection. This has the property that all elements of S are mapped to units, because $S \cap \mathfrak{p} = \varnothing$. By the universal property of the localisation, we have a map

$$\varphi : S^{-1}R \to FF(R/\mathfrak{p}); \quad \frac{r}{s} \mapsto \frac{r + \mathfrak{p}}{s + \mathfrak{p}}$$

It suffices to show that $\ker\varphi = \mathfrak{p}^e$, then the result holds by the isomorphism theorem. Let $\frac{r}{s} \in \ker\varphi$, so $\frac{r+\mathfrak{p}}{s+\mathfrak{p}} = \frac{0}{1}$ in $FF(R/\mathfrak{p})$. Observe that $\operatorname{im}\varphi \subseteq \overline{S}^{-1}(R/\mathfrak{p})$, where $\overline{S}$ is the image of S in $R/\mathfrak{p}$. Restricting the range, we can consider φ as a map from $S^{-1}R$ to $\overline{S}^{-1}(R/\mathfrak{p})$. So $\varphi\left(\frac{r}{s}\right) = \frac{0}{1}$ implies that there exists $u + \mathfrak{p} \in \overline{S}$ such that $(u + \mathfrak{p})(r + \mathfrak{p}) = 0$, so $ur + \mathfrak{p} = 0$. In particular, $u \in S$ and $ur \in \mathfrak{p}$. Hence $\frac{r}{s} = \frac{ur}{us}$ where $ur \in \mathfrak{p}$ and $us \in S$, so $\frac{r}{s} \in \mathfrak{p}^e$.

For the other direction, take $x \in \mathfrak{p}^e$, so $x = \frac{p}{s}$ for $p \in \mathfrak{p}, s \in S$. Then $\varphi(x) = \frac{p+\mathfrak{p}}{s+\mathfrak{p}} = 0$, so $x \in \ker\varphi$. □

It is not true in general that the extensions of prime ideals are prime.

Definition. If I is an ideal in R, the *radical* of I is the ideal

$$\sqrt{I} = \{r \in R \mid \exists n \geq 1,\ r^n \in I\}$$

Proposition. Let I be an ideal in a ring R. Then

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$$

Proof. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some $n \geq 1$. For every $\mathfrak{p} \in \operatorname{Spec} R$, if $I \subseteq \mathfrak{p}$, then $x^n \in \mathfrak{p}$, so $x \in \mathfrak{p}$. Conversely, suppose $x^n \notin I$ for all $n \geq 1$. As $I \neq R$, we have ${}^{R}\!/\!_{I} \neq 0$. Let $\overline{x}$ be the image of x in ${}^{R}\!/\!_{I}$, and consider

$$\left({}^{R}\!/\!_{I}\right)_{\overline{x}} = \left\{\overline{x}^n \,\middle|\, n \geq 1\right\}^{-1}\left({}^{R}\!/\!_{I}\right)$$

This is not the zero ring, because $x^n \notin I$ for all $n \geq 1$. Therefore, $\left({}^{R}\!/\!_{I}\right)_{\overline{x}}$ has a prime ideal, as it contains a maximal ideal. By the bijection described in part (iv) of the previous result, this prime ideal corresponds to a prime ideal of ${}^{R}\!/\!_{I}$ that avoids $\overline{x}$. This in turn corresponds to a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ that contains I and avoids x. Hence $x \notin \bigcap_{I \subseteq \mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$. □

3.7 Local properties

Definition. A ring R is *local* if it has exactly one maximal ideal.

We write $\operatorname{mSpec} R$ for the set of maximal ideals of R.

Example. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then there is a bijection between the prime ideals of R contained within $\mathfrak{p}$ to $\operatorname{Spec} R_{\mathfrak{p}}$, mapping $\mathfrak{n} \mapsto \mathfrak{n}R_{\mathfrak{p}}$ and $\mathfrak{q}^c \mapsfrom \mathfrak{q}$. Hence, all prime ideals of $R_{\mathfrak{p}}$ are contained in $\mathfrak{p}^e = \mathfrak{p}R_{\mathfrak{p}}$. Thus $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ is a local ring.

Example. Recall that

$$\mathbb{Z}_{(2)} = \left\{\frac{a}{b} \,\middle|\, a, b \in \mathbb{Z},\ 2 \nmid b\right\}$$

This ring is local, and the unique maximal ideal is

$$(2)\mathbb{Z}_{(2)} = \left\{\frac{2a}{b} \,\middle|\, a, b \in \mathbb{Z},\ 2 \nmid b\right\}$$

Proposition. Let M be an R-module. The following are equivalent.

(i) M is the zero module;

(ii) $M_{\mathfrak{p}}$ is the zero module for all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$;

(iii) $M_{\mathfrak{m}}$ is the zero module for all maximal ideals $\mathfrak{m} \in \operatorname{mSpec} R$.

Informally, for modules, being zero is a local property.

Proof. First, note that (i) implies (ii) and (ii) implies (iii). We show that (iii) implies (i). Suppose that M is not the zero module, so let $m \in M$ be a nonzero element. Consider $\mathrm{Ann}_R(m) = \{r \in R \mid rm = 0\}$. This is an ideal of R, but is a proper ideal because $1 \notin \mathrm{Ann}_R(m)$. Let $\mathfrak{m}$ be a maximal ideal of R containing $\mathrm{Ann}_R(m)$. Now, $\frac{m}{1} \in M_\mathfrak{m} = 0$. Thus, $\frac{m}{1} = \frac{0}{1}$, so $um = 0$ for some $u \in R \setminus \mathfrak{m}$. But then $u \notin \mathrm{Ann}_R(m)$, giving a contradiction. □

Proposition. Let $f : M \to N$ be an R-linear map. The following are equivalent.
(i) f is injective;
(ii) $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$;
(iii) $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is injective for every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} R$.

The same result holds for surjectivity.

Proof. The fact that (i) implies (ii) follows directly from the fact that localisation at $\mathfrak{p}$ is an exact functor. Clearly (ii) implies (iii). Suppose that $f_\mathfrak{m}$ is injective for each $\mathfrak{m} \in \operatorname{mSpec} R$. We have the following exact sequence.

$$0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N$$

As $(-)_\mathfrak{p}$ is exact, the sequence

$$0 \longrightarrow (\ker f)_\mathfrak{m} \longrightarrow M_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} N_\mathfrak{m}$$

is exact. But by assumption, $(\ker f)_\mathfrak{m} = \ker(f_\mathfrak{m}) = 0$. So $(\ker f)_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m} \in \operatorname{mSpec} R$, so $\ker f = 0$. □

Proposition. Let M be an R-module. The following are equivalent.
(i) M is a flat R-module;
(ii) $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$;
(iii) $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m} \in \operatorname{mSpec} R$.

Proof. (i) implies (ii). Note that $M_\mathfrak{p} \simeq R_\mathfrak{p} \otimes_R M$ as $R_\mathfrak{p}$-modules, by extension of scalars. Since extension of scalars preserves flatness, $M_\mathfrak{p}$ is flat.

Clearly (ii) implies (iii).

(iii) implies (i). Let $f : N \to P$ be an R-linear injective map. Let $\mathfrak{m} \in \operatorname{mSpec} R$. Then $f_\mathfrak{m} : N_\mathfrak{m} \to P_\mathfrak{m}$ is injective by the previous proposition. Note that the following diagram commutes.

$$\begin{tikzcd}
N_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m} \arrow[r, "f_\mathfrak{m} \otimes \mathrm{id}_{M_\mathfrak{m}}"] \arrow[d, "\sim"'] & P_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m} \arrow[d, "\sim"] \\
(N \otimes_R M)_\mathfrak{m} \arrow[r, "(f \otimes \mathrm{id}_M)_\mathfrak{m}"'] & (P \otimes_R M)_\mathfrak{m}
\end{tikzcd}$$

Hence $(f \otimes \mathrm{id}_M)_\mathfrak{m}$ is injective. Since this holds for each $\mathfrak{m} \in \operatorname{mSpec} R$, the map $f \otimes \mathrm{id}_M$ must be injective, as required. □

Example. An R-module M is *locally free* if $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for every prime ideal $\mathfrak{p} \in \operatorname{Spec} R$. Consider $R = \mathbb{C} \otimes \mathbb{C}$. Then

$$\operatorname{Spec} R = \{\mathfrak{p} \times \mathbb{C} \mid \mathfrak{p} \in \operatorname{Spec} \mathbb{C}\} \cup \{\mathbb{C} \times \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} \mathbb{C}\} = \{\mathbb{C} \times (0), (0) \times \mathbb{C}\}$$

The map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by $(a, b) \mapsto b$ sends $(\mathbb{C} \times \mathbb{C}) \setminus \mathbb{C} \times (0)$ to units. Thus, by the universal property of the localisation, we have a map

$$(\mathbb{C} \times \mathbb{C})_{\mathbb{C} \times (0)} \to \mathbb{C}; \quad \frac{(a, b)}{(c, d)} \mapsto \frac{b}{d}$$

This is clearly surjective, and one can check that this is also injective. Thus $(\mathbb{C} \times \mathbb{C})_{\mathbb{C} \times (0)} \simeq \mathbb{C}$ is a field. Similarly, $(\mathbb{C} \times \mathbb{C})_{(0) \times \mathbb{C}}$ is a field. So for every $\mathbb{C} \times \mathbb{C}$-module M and prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$, the module $M_\mathfrak{p}$ is a $\mathbb{C}$-vector space, so is free. Thus every module over $\mathbb{C} \times \mathbb{C}$ is locally free, but not every module over $\mathbb{C} \times \mathbb{C}$ is free. For example, take $M = \mathbb{C} \times \{0\}$ as a $\mathbb{C} \times \mathbb{C}$-module. One can show that M is not the zero module, and not free of rank at least 1, so cannot be free.

3.8 Localisations as quotients

Let $U \subseteq R$, and let $S \subseteq R$ be its multiplicative closure. We can define

$$R_U = {}^{R[\{T_u\}_{u \in U}]}\big/_{I_U}; \quad I_U = \left(\{uT_u - 1\}_{u \in U}\right)$$

We claim that $R_U = S^{-1}R$ as rings, and also as R-algebras. Writing $\overline{u}$ and $\overline{T}_u$ to be the images of these elements in R_U, the isomorphism maps

$$\overline{T}_u \mapsto \frac{1}{u}; \quad rT_{u_1} \dots T_{u_\ell} + I_U \leftarrow\!\shortmid \frac{r}{u_1 \dots u_\ell}$$

This is because R_U has the universal property of $S^{-1}R$. Indeed, for any $f : R \to A$ mapping U to units, there is a unique h making the following diagram commute.

$$\begin{tikzcd} R \arrow[r] \arrow[dr, "f"'] & R_U \arrow[d, dashed, "h"] \\ & A \end{tikzcd}$$

Note that A is an R-algebra via f, so the diagram commutes if and only if h is an R-algebra homomorphism. We have

$$\operatorname{Hom}_{R\text{-algebra}}(R_U, A) \simeq \{\varphi : U \to A \mid f(u)\varphi(u) = 1\}$$

But the the right hand side is a singleton.

Example. Let $x \in R$, and consider $R_x = R_{\{1, x, x^2, \dots\}}$. Here,

$$R_x \simeq {}^{R[T]}\big/_{(xT - 1)}$$

4 Integrality, finiteness, and finite generation

4.1 Nakayama's lemma

Proposition (Cayley–Hamilton theorem)**.** Let M be a finitely generated R-module, and let $f : M \to M$ be an R-linear endomorphism. Let $\mathfrak{a}$ be an ideal in R such that $f(M) \subseteq \mathfrak{a}M$. Then, we have an equality in $\operatorname{End}_R M$

$$f^n + a_1 f^{n-1} + \dots + a_n f^0 = 0; \quad f^r = \underbrace{f \circ \dots \circ f}_{r \text{ times}}$$

where $a_i \in \mathfrak{a}$.

Proof. Let $M = \operatorname{span}_R \{m_1, \dots, m_n\}$, so $\mathfrak{a}M = \operatorname{span}_{\mathfrak{a}} \{m_1, \dots, m_n\}$. Then

$$\begin{pmatrix} f(m_1) \\ \vdots \\ f(m_n) \end{pmatrix} = P \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}; \quad P \in M_{n\times n}(\mathfrak{a})$$

Let $\rho : R \to \operatorname{End} M$ be the structure ring homomorphism of M as an R-module. Then we can define $R[T] \to \operatorname{End} M$ by $T \mapsto f$, making M into an $R[T]$-module. Hence,

$$T \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = P \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

Thus

$$Q \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0; \quad Q = TI_n - P$$

Multiplying by the adjugate matrix $\operatorname{adj} Q$ on the left on both sides,

$$(\det Q) \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

In particular, $(\det Q)m = 0$ for all $m \in M$, as the m_i generate M. Hence, $m \mapsto (\det Q)m = (\det Q)|_{T=f}$ is 0 in $\operatorname{End}_R M$. Finally, note that $\det Q$ is a monic polynomial, and all other coefficients lie in $\mathfrak{a}$. □

Corollary. Let M be a finitely generated R-module, and let $\mathfrak{a}$ be an ideal in R. If $\mathfrak{a}M = M$, then there exists $a \in \mathfrak{a}$ such that $am = m$ for all $m \in M$.

Proof. Apply the Cayley–Hamilton theorem with $f = \mathrm{id}_M$. We obtain a polynomial

$$(1 + a_1 + \dots + a_n)\,\mathrm{id}_M = 0$$

Take $a = -(a_1 + \dots + a_n)$. □

Definition. The *Jacobson radical* of a ring R, denoted $J(R)$, is the intersection of all maximal ideals of R.

Example. (i) If $(R, \mathfrak{m})$ is a local ring, then $J(R) = \mathfrak{m}$.

(ii) $J(\mathbb{Z}) = \{0\}$.

Proposition. Let $x \in R$. Then $x \in J(R)$ if and only if $1 - xy$ is a unit for every $y \in R$.

Proof. First, let $x \in J(R)$, and suppose $y \in R$ is such that $1 - xy$ is not a unit. Then $(1 - xy)$ is a proper ideal, so it is contained in a maximal ideal $\mathfrak{m}$. But as $x \in J(R)$, we must have $x \in \mathfrak{m}$, giving $1 = 1 - xy + xy \in \mathfrak{m}$, contradicting that $\mathfrak{m}$ is a maximal ideal.

Now suppose $x \notin J(R)$, so there is a maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$. Then $\mathfrak{m} + (x) = R$ as $\mathfrak{m}$ is maximal. In particular, there exists $t \in \mathfrak{m}$ and $y \in R$ such that $t + xy = 1$, or equivalently, $1 - xy = t \in \mathfrak{m}$. Note that t cannot be a unit, because it is contained in a proper ideal. $\square$

Proposition (Nakayama's lemma)**.** Let M be a finitely generated R-module, and let $\mathfrak{a} \subseteq J(R)$ be an ideal of R such that $\mathfrak{a}M = M$. Then $M = 0$.

This lemma is more useful when $J(R)$ is large, so is particularly useful when applied to local rings.

Proof. By the above corollary, there exists $a \in \mathfrak{a}$ such that $am = m$ for all $m \in M$, or equivalently, $(1 - a)m = 0$. By assumption, $a \in J(R)$, so $1 - a$ is a unit in R. Hence $m = 0$. $\square$

Corollary. Let M be a finitely generated R-module, and let $N \subseteq M$ be a submodule. Let $\mathfrak{a} \subseteq J(R)$ be an ideal in R such that $N + \mathfrak{a}M = M$. Then $N = M$.

This can be applied to find generating sets for M.

Proof. Note that

$$\mathfrak{a}\left(M/N\right) = (\mathfrak{a}M + N)/N = M/N$$

so $M/N = 0$ by Nakayama's lemma. $\square$

4.2 Integral and finite extensions

Definition. Let A be an R-algebra, and let $x \in A$. Then x is *integral* over R if there exists a monic polynomial $f \in R[T]$ such that $f(x) = 0$.

Example. (i) If $R = k$ is a field, then x is integral over k if and only if x is algebraic over k.

(ii) We will prove later that

(a) the $\mathbb{Z}$-integral elements of $\mathbb{Q}$ are $\mathbb{Z}$;

(b) the $\mathbb{Z}$-integral elements of $\mathbb{Q}\left[\sqrt{2}\right]$ are $\mathbb{Z}\left[\sqrt{2}\right]$;

(c) the $\mathbb{Z}$-integral elements of $\mathbb{Q}\left[\sqrt{5}\right]$ are $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right] \supsetneq \mathbb{Z}\left[\sqrt{5}\right]$.

Definition. Let M be an R-module. We say that M is *faithful* if the structure homomorphism $\rho : R \to \operatorname{End} M$ is injective. Equivalently, for every nonzero ring element r, there exists

$m \in M$ such that $rm \neq 0$.

Example. Let $R \subseteq A$ be rings, and let A be an R-module in the natural way. Then A is a faithful R-module, as if $r \neq 0$, then $r1_A = r \neq 0$.

Proposition. Let $R \subseteq A$ be rings and $x \in A$, and consider A as an $R[x]$-module. Then x is integral over R if and only if there exists $M \subseteq A$ such that
(i) M is a faithful $R[x]$-module; and
(ii) M is finitely generated as an R-module.

Condition (i) is that M is an R-submodule of A, $xM \subseteq M$, and M is faithful over $R[x]$.

Proof. First, assume conditions (i) and (ii) hold. We have an R-linear map $f : M \to M$ given by multiplication by x, as $xM \subseteq M$. As M is a finitely generated R-module, we can apply the Cayley–Hamilton theorem to find

$$f^n + r_1 f^{n-1} + \dots + r_n f^0 = 0; \quad r_i \in R$$

in $\operatorname{End}_R M$. Then, evaluating at $m \in M$,

$$(x^n + r_1 x^{n-1} + \dots + r_n x^0)m = 0$$

As this holds for all m, and M is a faithful $R[x]$-module, we must have

$$x^n + r_1 x^{n-1} + \dots + r_n x^0 = 0$$

Thus x is integral over R.

Now suppose x is integral over R. Then

$$x^n + r_1 x^{n-1} + \dots + r_n x^0 = 0$$

for some $r_1, \dots, r_n \in R$. We define

$$M = \operatorname{span}_R \left\{ x_0, \dots, x^{n-1} \right\}$$

This is finitely generated, and satisfies $xM \subseteq M$. M is faithful over $R[x]$ as it contains $x^0 = 1$. $\square$

Definition. Let A be an R-algebra. Then A is
(i) *integral* over R, if all of its elements are integral over R;
(ii) *finite* over R, if A is finitely generated as an R-module.

Proposition. Let A be an R-algebra. Then the following are equivalent.
(i) A is a finitely generated R-algebra and is integral over R;
(ii) A is generated as an R-algebra by a finite set of integral elements;
(iii) A is finite over R.

Proof. (i) implies (ii). The generators for A are integral.

(ii) implies (iii). Suppose A is generated by $\alpha_1, \dots, \alpha_m$ as an R-algebra, and the α_i are integral over R. As α_i is integral,

$$\alpha_i^{n_i} + r_{i,1}\alpha_i^{n_i-1} + \dots + r_{i,n_i}\alpha_i^0 = 0$$

Hence $\alpha_i^{n_i}$ lies in the R-linear span of $\left\{\alpha_i^0, \dots, \alpha_i^{n_i-1}\right\}$. Thus, every element is an R-linear combination of products of the form $\alpha_1^{e_1} \dots \alpha_n^{e_n}$, which in turn lies in the R-linear span of products of the same form where all e_i are less than the corresponding n_i. This is a finite set, so A is finitely generated as an R-module.

(iii) implies (i). As A is finitely generated as an R-module, it must be finitely generated as an R-algebra. Let $\alpha \in A$; we show α is integral over R. Let $\rho : R \to A$ be the structure homomorphism of A as an R-algebra. Then $\rho(R) \subseteq A$, and consider $(\rho(R))[\alpha] \subseteq A$. Now, A is a $(\rho(R))[\alpha]$-module, and is faithful because $1_A \in A$. As A is a finitely generated $\rho(R)$-module, the previous proposition shows that α is $\rho(R)$-integral. Equivalently, α is R-integral. □

Proposition. Let A be an R-algebra and let $\mathcal{O}$ be the set of elements of A that are integral over R. Then $\mathcal{O}$ is an R-subalgebra of A.

Proof. Let $x, y \in \mathcal{O}$. Then $\{x, y\}$ is a finite set of R-integral elements, so the set generates an integral R-subalgebra of A. Hence $x + y, xy$ lie in this subalgebra, and so they are integral. □

Proposition. Let $A \subseteq B \subseteq C$ be rings. Then,

(i) if C is finite over B and B is finite over A, then C is finite over A;
(ii) if C is integral over B and B is integral over A, then C is integral over A.

Proof. Part (i). Suppose that

$$C = \operatorname{span}_B \{\gamma_1, \dots, \gamma_n\}; \quad B = \operatorname{span}_A \{\beta_1, \dots, \beta_\ell\}$$

Then

$$C = \operatorname{span}_A \left\{\gamma_i\beta_j \mid i \leq n, j \leq \ell\right\}$$

Part (ii). Let $c \in C$, so $f(c) = 0$ for

$$f(T) = T^n + b_1T^{n-1} + \dots + b_nT^0 \in B[T]$$

Then $f \in A'[T]$, where $A' = A[b_1, \dots, b_n]$. The inclusion $A \subseteq A'$ is generated as an A-algebra by finitely many integral elements. Similarly, $A' \subseteq A'[c]$ is generated as an A-algebra by c, which is integral over A' as $f \in A'[T]$. By the previous result, both extensions are finite. Then, by part (i), $A \subseteq A'[c]$ is finite, so c is integral over A. □

4.3 Integral closure

Definition. Let $A \subseteq B$ be rings. The *integral closure* of A in B is the set $\overline{A}$ of elements of B that are integral over A, which is an A-algebra. We say that A is *integrally closed* in B if $\overline{A} = A$.

Definition. Let A be an integral domain. In this case, the *integral closure* of A is the integral closure of A in its field of fractions $FF(A)$. We say that A is integrally closed if it is integrally closed in its field of fractions.

Example. (i) $\mathbb{Z}\left[\sqrt{5}\right]$ is not integrally closed, because $\alpha = \frac{1+\sqrt{5}}{2} \in FF\left(\mathbb{Z}\left[\sqrt{5}\right]\right) = \mathbb{Q}\left[\sqrt{5}\right]$, and $\alpha^2 - \alpha - 1 = 0$ so it is $\mathbb{Z}\left[\sqrt{5}\right]$-integral.

(ii) $\mathbb{Z}$ is integrally closed.

(iii) If k is a field, $k[T_1, \dots, T_n]$ are integrally closed.

Examples (ii) and (iii) are special cases of the following result.

Proposition. Let A be a unique factorisation domain. Then A is integrally closed.

Proof. Let $x \in FF(A) \setminus A$, and write $x = \frac{a}{b}$ with $a \in A, b \in A \setminus \{0\}$. As A is a unique factorisation domain, we can assume there is a prime p such that $p \mid b$ and $p \nmid a$. If x is integral over A, then

$$\left(\frac{a}{b}\right)^n + a_1\left(\frac{a}{b}\right)^{n-1} + \dots + a_n\left(\frac{a}{b}\right)^0 = 0$$

Multiplying by b^n,

$$a^n = -b\left(a_1 b_0 a^{n-1} + \dots + a_n b^{n-1} a^0\right)$$

But as $p \mid b$, we must have $p \mid a^n$, so $p \mid a$, which is a contradiction. □

Lemma. Let $A \subseteq B$ be rings, and let $\overline{A}$ be the integral closure of A in B. Then $\overline{A}$ is integrally closed in B.

Taking the integral closure is an idempotent operation.

Proof. Let $x \in B$ be integral over $\overline{A}$. Then, we have

$$A \subseteq \overline{A} \subseteq \overline{A}[x]$$

The first extension is integral by definition, and the second is integral by the above proposition, as x is integral over $\overline{A}$. By transitivity of integrality, $\overline{A}[x]$ is integral over A, so in particular, x is integral over A. Thus $x \in \overline{A}$. □

Proposition. Let $A \subseteq B$ be rings.

(i) if B is integral over A and $\mathfrak{b}$ is an ideal in B, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{b}^c$;

(ii) if B is integral over A and $S \subseteq A$ is a multiplicative set, then $S^{-1}B$ is integral over $S^{-1}A$;
(iii) if $\overline{A}$ is the integral closure of A in B and $S \subseteq A$ is a multiplicative set, then $S^{-1}\overline{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$, so $\overline{S^{-1}A} = S^{-1}\overline{A}$.

The proofs follow directly from the definitions.

Lemma. Let $A \subseteq B$ be an integral extension of rings. Then
(i) $A \cap B^\times = A^\times$;
(ii) if A, B are integral domains, then A is a field if and only if B is a field.

Proof. Part (i). One inclusion is clear: $A^\times \subseteq A \cap B^\times$. Suppose $a \in A$ and a is a unit in B with inverse $b \in B$; we show that $b \in A$. As b is integral over A,

$$b^n + a_1 b^{n-1} + \dots + a_n b^0 = 0; \quad a_i \in A$$

Multiplying by a^{n-1},

$$b + \underbrace{a_1 + a_2 a^1 + \dots + a_n a^{n-1}}_{\in A} = 0$$

Hence b must lie in A.

Part (ii). Suppose B is a field. Then

$$A^\times = A \cap (B \setminus \{0\}) = A \setminus \{0\}$$

Hence A is a field. Conversely, suppose A is a field. Let $b \in B$ be a nonzero element; we want to show that b is a unit in B. As b is integral over A,

$$b^n + a_1 b^{n-1} + \dots + a_n b^0 = 0; \quad a_i \in A$$

Let n be minimal with this property. Then

$$b\,\underbrace{(b^{n-1} + a_1 b^{n-2} + \dots + a_{n-1} b^0)}_{\Delta} = -a_n$$

Note that $b \neq 0$ by assumption, and $\Delta \neq 0$ by minimality. As B is an integral domain, $a_n \neq 0$. Because A is a field, a_n is invertible. Thus

$$b(-a_n^{-1}\Delta) = 1 \implies b \in B^\times$$

□

Corollary. Let $A \subseteq B$ be an integral extension of rings, and let $\mathfrak{q}$ be a prime ideal in B. Then $\mathfrak{q}$ is a maximal ideal of B if and only if it $\mathfrak{q}^c = \mathfrak{q} \cap A$ is a maximal ideal in A.

Proof. We have an embedding of rings

$$A/_{\mathfrak{q} \cap A} \rightarrowtail B/_{\mathfrak{q}}$$

which is an integral extension of integral domains. By the previous result, one is a field if and only if the other is, so $\mathfrak{q} \cap A$ is maximal in A if and only if $\mathfrak{q}$ is maximal in B. □

4.4 Noether normalisation

Definition. Let A be a k-algebra, and let $x_1, \dots, x_n \in A$. We say that $x_1, \dots, x_n$ are *k-algebraically independent* if for every nonzero polynomial $p \in k[T_1, \dots, T_n]$, we have $p(x_1, \dots, x_n) \neq 0$. Equivalently, the k-algebra homomorphism $k[T_1, \dots, T_n] \to A$ given by $T_i \mapsto x_i$ is injective.

Theorem (Noether's normalisation theorem)**.** Let k be a field, and let $A \neq 0$ be a finitely generated k-algebra. Then there exist $x_1, \dots, x_n \in A$ which are k-algebraically independent and A is finite over $A' = k[x_1, \dots, x_n]$.

We first present an example of the method used in the proof.

Example. Let $A = k[T, T^{-1}] \simeq k[X, Y]/(XY - 1)$. We claim that $k[T] \subseteq k[T, T^{-1}]$ is not a finite extension. Indeed, suppose it were finite. Then T^{-1} would be integral over $k[T]$, so

$$(T^{-1})^n \in \operatorname{span}_{k[T]} \left\{(T^{-1})^0, \dots, (T^{-1})^{n-1}\right\}$$

Multiplying by T^n, we have

$$1 \in \operatorname{span}_{k[T]}(T^n, \dots, T)$$

which is false. However, if $c \in k$ is a scalar which we will choose later,

$$A = k[T, T^{-1}] = k[T, T^{-1} - cT]$$

We claim that $k[T^{-1} - cT] \subseteq A$ is a finite extension for most values of c, and in particular, for at least one. First, note $T^{-1}T - 1 = 0$, and then change variables to

$$((T^{-1} - cT) + cT)T - 1 = 0 \implies \underbrace{c}_{\in k} T^2 + \underbrace{(T^{-1} - cT)}_{\in k[T^{-1} - ct]} T - \underbrace{1}_{\in k[T^{-1} - cT]} = 0$$

Hence if $c \neq 0$, T is integral over $k[T^{-1} - cT]$.

Proof. In this proof, we will assume k is infinite, although the theorem is true even if k if finite. We will proceed by induction on the minimal number of generators of A as a k-algebra, which we will denote m. For the case $m = 0$, we have $A = k$, so we can take $A' = k$.

Suppose that A is generated as a k-algebra by $x_1, \dots, x_m \in A$. If $x_1, \dots, x_m$ are algebraically independent, then we can take $A' = A$. Otherwise, we claim that there are $c_1, \dots, c_{m-1} \in k$ such that x_m is integral over

$$B = k[x_1 - c_1 x_m, \dots, x_{m-1} - c_{m-1} x_m]$$

Assuming that this holds, we have $A = B[x_m]$, so $B \subseteq A$ is a finite extension. But B is generated by $m - 1$ elements, so by induction B contains $z_1, \dots, z_n \in B$ which are k-algebraically independent, and B is finite over $A' = k[z_1, \dots, z_n]$. Then A is finite over A' by transitivity of finiteness.

We now prove the claim. As $x_1, \dots, x_m$ are not algebraically independent over k, there is a nonzero polynomial $f \in k[T_1, \dots, T_m]$ such that $f(x_1, \dots, x_m) = 0$. We want to show that x_m is integral over B. Write f as the sum of its homogeneous parts, and let F be the part of highest degree $\deg f = r$. For scalars $c_1, \dots, c_{m-1} \in k$ which will be chosen later, we define

$$\begin{aligned} g(T_1, \dots, T_m) &= f(T_1 + c_1 T_m, \dots, T_{m-1} + c_{m-1} T_m, T_m) \\ &= \underbrace{F(c_1, \dots, c_m, 1)}_{\in k} T_m^r + \text{terms of lower degree in } T_m \text{ with coefficients in } k[T_1, \dots, T_{m-1}] \end{aligned}$$

Note that

$$g(x_1 - c_1x_m, \dots, x_{m-1} - c_{m-1}x_m, x_m) = f(x_1, \dots, x_m) = 0$$

but as a polynomial in T_m over $k[T_1, \dots, T_{m-1}]$, it has degree at most r, and the coefficient of T_m^r is $F(c_1, \dots, c_m, 1)$. As $F(T_1, \dots, T_m)$ is a nonzero homogeneous polynomial, $F(T_1, \dots, T_{m-1}, 1)$ is not the zero polynomial. Thus there are $c_1, \dots, c_{m-1}$ such that $F(c_1, \dots, c_{m-1}, 1) \neq 0$ as k is an infinite field. □

4.5 Hilbert's Nullstellensatz

Proposition (Zariski's lemma)**.** Let $k \subseteq L$ be fields where L is finitely generated as a k-algebra. Then $\dim_k L$ is finite.

Proof. By Noether normalisation, we have

$$k \subseteq k[x_1, \dots, x_n] \subseteq L$$

where $x_1, \dots, x_n$ are algebraically independent over k, and L is finite over $k[x_1, \dots, x_n]$. As this is an integral extension of integral domains and L is a field, $k[x_1, \dots, x_n]$ must be a field. But as $k[x_1, \dots, x_n]$ is a polynomial algebra over k, the x_i cannot be invertible. Hence $n = 0$, so $k \subseteq L$ is finite as required. □

Definition. Let $k \subseteq \Omega$ be an extension of fields, where Ω is algebraically closed.

(i) Let $S \subseteq k[T_1, \dots, T_n]$. We define

$$\mathbb{V}(S) = \{\mathbf{x} \in \Omega^n \mid \forall f \in S,\ f(\mathbf{x}) = 0\}$$

Sets of this form are called *k-algebraic* subsets of Ω^n.

(ii) Let $X \subseteq \Omega^n$. We define

$$I(X) = \{f \in k[T_1, \dots, T_n] \mid \forall \mathbf{x} \in X,\ f(\mathbf{x}) = 0\}$$

Note that $\mathbb{V}(S) = \mathbb{V}(I)$, where I is the ideal generated by S. Recall that for every finite field extension $k \subseteq L$, there is a k-algebra embedding $L \to \Omega$, because Ω is algebraically closed.

Theorem. Let $\mathfrak{a} \subseteq k[T_1, \dots, T_n]$ be an ideal. Then

(i) (weak Nullstellensatz) $\mathbb{V}(\mathfrak{a}) = \varnothing$ if and only if $1 \in \mathfrak{a}$;

(ii) (strong Nullstellensatz) $I(\mathbb{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Proof. Weak Nullstellensatz. Clearly if $1 \in \mathfrak{a}$ then $\mathbb{V}(\mathfrak{a}) = \varnothing$, as $1 \neq 0$. Now suppose $1 \notin \mathfrak{a}$. There is a maximal ideal $\mathfrak{m} \in \operatorname{mSpec} k[T_1, \dots, T_n]$ such that $\mathfrak{a} \subseteq \mathfrak{m}$. Then $L = {}^{k[T_1, \dots, T_n]}\!/\!_{\mathfrak{m}}$ is a field, which is finitely generated over k as an algebra. By Zariski's lemma, this extension is finitely generated as a module. Hence, there is an injective k-algebra homomorphism $L \to \Omega$. Composing with the quotient map, we obtain a k-algebra homomorphism $\varphi : k[T_1, \dots, T_n] \to \Omega$ with kernel $\mathfrak{m}$. Now, let

$$\mathbf{x} = (\varphi(T_1), \dots, \varphi(T_n)) \in \Omega^n$$

We claim that this is a common solution to all polynomials in $\mathfrak{a}$. Note that for $f \in k[T_1, \dots, T_n]$, we have $\varphi(f) = f(\mathbf{x})$. Therefore, for all $f \in \mathfrak{a}$, we have $f \in \ker\varphi$ so $f(\mathbf{x}) = \varphi(f) = 0$.

Strong Nullstellensatz. Let $f \in \sqrt{\mathfrak{a}}$. Then $f^\ell \in \mathfrak{a}$ for some $\ell \geq 1$, and therefore, $f^\ell(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{V}(\mathfrak{a})$. As Ω is an integral domain, $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{V}(\mathfrak{a})$. Hence $f \in I(\mathbb{V}(\mathfrak{a}))$.

Conversely, suppose $f \in I(\mathbb{V}(\mathfrak{a}))$, so for all $\mathbf{x} \in \mathbb{V}(\mathfrak{a})$, we have $f(\mathbf{x}) = 0$. We want to show that $f \in \sqrt{\mathfrak{a}}$. To do this, we show that $\overline{f}$ is nilpotent in $k[T_1, \dots, T_n]/\mathfrak{a}$. It suffices to show that

$$\left(k[T_1, \dots, T_n]/\mathfrak{a}\right)_{\overline{f}} = 0$$

Note that

$$\left(k[T_1, \dots, T_n]/\mathfrak{a}\right)_{\overline{f}} \simeq k[T_1, \dots, T_n, T_{n+1}]/\mathfrak{b}; \quad \mathfrak{b} = \mathfrak{a}^e + (T_{n+1}f - 1)$$

We will show that $1 \in \mathfrak{b}$, or equivalently by the weak Nullstellensatz, $\mathbb{V}(\mathfrak{b}) = \varnothing$.

Suppose $\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{V}(\mathfrak{b}) \subseteq \Omega^{n+1}$. Define $\mathbf{x}_0 = (x_1, \dots, x_n)$, so $\mathbf{x}_0 \in \mathbb{V}(\mathfrak{a})$. In particular, $f(\mathbf{x}_0) = 0$, as $f \in I(\mathbb{V}(\mathfrak{a}))$. Thus $f(\mathbf{x}) = 0$. Now, $(T_{n+1}f - 1)(\mathbf{x}) = -1 \neq 0$, but $(T_{n+1}f - 1) \in \mathfrak{b}$, so $\mathbf{x}$ is not a common solution to all polynomials in $\mathfrak{b}$, which is a contradiction. $\square$

One can easily derive the weak Nullstellensatz from the strong Nullstellensatz.

Note that

(i) $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.

(ii) If $X \subseteq Y \subseteq \Omega^n$, then $I(X) \supseteq I(Y)$.

(iii) If $S \subseteq T \subseteq k[T_1, \dots, T_n]$, then $\mathbb{V}(S) \supseteq \mathbb{V}(T)$.

(iv) If $S \subseteq k[T_1, \dots, T_n]$, then $S \subseteq I(\mathbb{V}(S))$.

(v) If $X \subseteq \Omega^n$, then $X \subseteq \mathbb{V}(I(X))$.

(vi) If $X \subseteq \Omega^n$ is an algebraic set, then $X = \mathbb{V}(I(X))$, as $X = \mathbb{V}(\mathfrak{a})$ gives

$$\mathbb{V}(\mathfrak{a}) \subseteq \mathbb{V}(I(\mathbb{V}(\mathfrak{a}))) \subseteq \mathbb{V}(\mathfrak{a})$$

(vii) If $X \subseteq \Omega^n$, then $I(X)$ is a radical ideal.

Proposition. Let $k = \Omega$ be an algebraically closed field, and let $n \geq 0$. Then we have an inclusion-reversing bijection

$$\{k\text{-algebraic subsets of } \Omega^n\} \leftrightarrow \{\text{radical ideals of } k[T_1, \dots, T_n]\}$$

given by $X \mapsto I(X)$ and $\mathbb{V}(\mathfrak{a}) \leftarrow\!\shortmid \mathfrak{a}$.

Proof. We have already shown that $I(X)$ is radical, and $X = \mathbb{V}(I(X))$ if X is an algebraic set. For the converse, let $\mathfrak{a} \subseteq k[T_1, \dots, T_n]$ be a radical ideal. Then $I(\mathbb{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$ by the strong Nullstellensatz. $\square$

Remark. Every prime ideal $\mathfrak{p}$ is radical, as $x^n \in \mathfrak{p}$ implies $x \in \mathfrak{p}$. In particular, every maximal ideal is radical.

Corollary. Let $k = \Omega$ be an algebraically closed field. Then we have a bijection

$$\Omega^n \leftrightarrow \operatorname{mSpec} k[T_1, \dots, T_n]$$

given by $\mathbf{x} = (x_1, \dots, x_n) \mapsto (T_1 - x_1, \dots, T_n - x_n) = \mathfrak{m}_{\mathbf{x}}$.

Proof. First, note that $\mathfrak{m}_{\mathbf{x}}$ is a maximal ideal for every $\mathbf{x}$, since it is the kernel of the map $k[T_1, \dots, T_n] \twoheadrightarrow \Omega$ given by $T_i \to x_i$. Also, $\mathfrak{m}_{\mathbf{x}} = I(\{\mathbf{x}\})$. Indeed, the inclusion $\mathfrak{m}_{\mathbf{x}} \subseteq I(\{\mathbf{x}\})$ is clear, and $I(\{\mathbf{x}\})$ is a proper ideal of $k[T_1, \dots, T_n]$, so they must be equal by maximality. Note that $\mathbb{V}(\mathfrak{m}_{\mathbf{x}}) = \{\mathbf{x}\}$. Hence the claim follows from the inclusion-reversing bijection, as maximal ideals correspond to minimal nonempty $k$algebraic sets. $\square$

Definition. We say that $X \subseteq \Omega^n$ is *irreducible* if X cannot be expressed as the union of two strictly smaller algebraic subsets.

Proposition. $X \subseteq \Omega^n$ is irreducible if and only if $I(X)$ is prime.

4.6 Integrality over ideals

Definition. Let $A \subseteq B$ be an extension of rings, and let $\mathfrak{a} \subseteq A$ be an ideal. We say that $x \in B$ is integral over $\mathfrak{a}$ if

$$x^n + a_1 x^{n-1} + \dots + a_n x^0 = 0$$

for some $a_1, \dots, a_n \in \mathfrak{a}$. The *integral closure* of $\mathfrak{a}$ in B is the set of elements of B that are integral over $\mathfrak{a}$.

Proposition. Let $A \subseteq B$ be an extension of rings, and let $\overline{A}$ be the integral closure of A in B. Let $\mathfrak{a}$ be an ideal of A. Then the integral closure of $\mathfrak{a}$ in B is $\sqrt{\mathfrak{a}\overline{A}}$, the radical in $\overline{A}$ of the extension of $\mathfrak{a}$ to $\overline{A}$.

Proof. If $b \in B$ is integral over $\mathfrak{a}$, then

$$b^n + a_1 b^{n-1} + \dots + a_n b^0 = 0; \quad a_i \in \mathfrak{a}$$

In particular, b lies in $\overline{A}$, and so all of its powers lie in $\overline{A}$ as $\overline{A}$ is a ring. Using the integrality equation for b, we observe that $b^n \in \mathfrak{a}\overline{A}$, hence $b \in \sqrt{\mathfrak{a}\overline{A}}$.

Now, suppose $b \in \sqrt{\mathfrak{a}\overline{A}}$. Then $b^n \in \mathfrak{a}\overline{A}$ for some n, so

$$b^n = \sum_{i=1}^m a_i x_i; \quad a_i \in \mathfrak{a}, x_i \in \overline{A}$$

Define $M = A[x_1, \dots, x_m]$. The generators lie in $\overline{A}$, so M is an A-algebra generated by finitely many integral elements over A. Hence M is a finite A-algebra. Note that $b^n M \subseteq \mathfrak{a}M$ by the equation for b^n, thought of as an extension of A-modules.

Now define $f : M \to M$ by multiplication by b^n. This satisfies $f(M) \subseteq \mathfrak{a}M$, and f is A-linear. Thus by the Cayley–Hamilton theorem,

$$f^\ell + \alpha_1 f^{\ell-1} + \dots + \alpha_\ell f^0 = 0 \in \operatorname{End}_R M; \quad \alpha_i \in \mathfrak{a}$$

Evaluating this at $1_A \in M$,

$$b^{n\ell} + \alpha_1 b^{n(\ell-1)} + \dots + \alpha_\ell b^0 = 0 \in B$$

This is an integrality relation for b is $\mathfrak{a}$-integral. □

Hence, the integral closure of an ideal is closed under sums and products.

Corollary. Let $A \subseteq B$ be an extension of rings, and let $\mathfrak{a}$ be an ideal of A. Then $b \in B$ is $\mathfrak{a}$-integral if and only if b is $\sqrt{\mathfrak{a}}$-integral.

Proof. By the previous proposition, it suffices to show that

$$\sqrt{\mathfrak{a}\overline{A}} = \sqrt{\sqrt{\mathfrak{a}}\,\overline{A}}$$

The forwards inclusion is clear. For the other direction, it is a general fact that $\sqrt{I}^e \subseteq \sqrt{I^e}$, so

$$\sqrt{\mathfrak{a}}A \subseteq \sqrt{\mathfrak{a}\overline{A}}$$

Taking radicals on both sides,

$$\sqrt{\sqrt{\mathfrak{a}}\,\overline{A}} \subseteq \sqrt{\sqrt{\mathfrak{a}\overline{A}}} = \sqrt{\mathfrak{a}\overline{A}}$$

□

Proposition. Let A be an integrally closed integral domain (in its field of fractions). Let $A \subseteq B$ be an extension of rings, let $\mathfrak{a}$ be an ideal in A, and let $b \in B$. The following are equivalent:

(i) b is integral over $\mathfrak{a}$;

(ii) b is algebraic over $FF(A)$ with minimal polynomial over $FF(A)$ of the form

$$T^n + a_1 T^{n-1} + \dots + a_n T^0 = 0; \quad a_i \in \sqrt{\mathfrak{a}}$$

Note that there is an embedding $FF(A) \subseteq FF(B)$.

Proof. Suppose (ii) holds. Then b is integral over $\sqrt{\mathfrak{a}}$ by definition. Thus, by the above corollary, b is integral over $\mathfrak{a}$.

Now suppose (i) holds. We have an integrality equation

$$b^n + a_1 b^{n-1} + \dots + a_n b^0 = 0; \quad a_i \in \mathfrak{a}$$

Define

$$h = T^n + a_1 T^{n-1} + \dots + a_n T^0 \in (FF(A))[T]$$

so $h(b) = 0$, so certainly b is algebraic over $FF(A)$. Let $f \in (FF(A))[T]$ be the minimal polynomial of b over $FF(A)$. Let $FF(A) \subseteq \Omega$ where Ω is an algebraically closed field, so

$$f = \prod_{i=1}^{\ell} (T - \alpha_i); \quad \alpha_1 = b, \alpha_i \in \Omega$$

We want to show that the coefficients of f are in $\sqrt{\mathfrak{a}}$. By the previous proposition, together with the fact that A is integrally closed, the integral closure of $\mathfrak{a}$ in $FF(A)$ is $\sqrt{\mathfrak{a}} \subseteq A$. So it suffices to show that the coefficients of f lie in $FF(A)$ and are integral over $\mathfrak{a}$. As f is the minimal polynomial over $FF(A)$, the first part holds by definition.

Expanding brackets in the equation for f, the coefficients of f are sums of products of the α_i. The proposition above implies that the integral closure of $\mathfrak{a}$ in Ω is closed under sums and products, so it suffices to show that the α_i are all integral over $\mathfrak{a}$. As the α_i and b have the same minimal polynomial f over $FF(A)$, there is an isomorphism of $FF(A)$-algebras $\varphi_i : FF(A)[b] \to FF(A)[\alpha_i]$ that maps b to α_i. Then as $h(b) = 0$ and $h \in (FF(A))[T]$, we must have $h(\alpha_i) = h(\varphi_i(b)) = \varphi_i(h(b)) = \varphi_i(0) = 0$. □

4.7 Cohen–Seidenberg theorems

If $A \subseteq B$ is an extension of rings, the inclusion $\iota : A \to B$ gives rise to $\iota^\star : \operatorname{Spec} B \to \operatorname{Spec} A$ given by $\iota(\mathfrak{q}) = \mathfrak{q} \cap A$. We will study the fibres of this induced map on spectra.

Proposition (incomparability). Let $A \subseteq B$ be an integral extension, and let $\mathfrak{q}, \mathfrak{q}'$ be prime ideals of B. Suppose that $\mathfrak{q}$ and $\mathfrak{q}'$ contract to the same prime ideal $\mathfrak{p} = \mathfrak{q} \cap A = \mathfrak{q}' \cap A$ of A, and that $\mathfrak{q} \subseteq \mathfrak{q}'$. Then $\mathfrak{q} = \mathfrak{q}'$.

We will write $B_\mathfrak{p}$ for $(A \setminus \mathfrak{p})^{-1}B$, but this is not in general a ring.

Proof. Define $S = A \setminus \mathfrak{p}$. Then $\mathfrak{q}$ and $\mathfrak{q}'$ are prime ideals of B not intersecting S. Hence $\mathfrak{q} = (S^{-1}\mathfrak{q})^c$, where $S^{-1}\mathfrak{q} = \mathfrak{q}B_\mathfrak{p}$ is the extension of $\mathfrak{q}$ to $S^{-1}B$, due to the bijection

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\} \leftrightarrow \operatorname{Spec} S^{-1}R$$

It suffices to show that $\mathfrak{q}B_\mathfrak{p} = \mathfrak{q}'B_\mathfrak{p}$, as then they are the contractions of the same ideal. Note that

$$\mathfrak{q}B_\mathfrak{p} \cap A_\mathfrak{p} = S^{-1}\mathfrak{q} \cap S^{-1}A = S^{-1}(\mathfrak{q} \cap A) = S^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$$

Similarly, $\mathfrak{q}'B_\mathfrak{p} \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, which is a maximal ideal of $A_\mathfrak{p}$. As $A \subseteq B$ is an integral extension, $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is also an integral extension. Recall that the contraction of a maximal ideal is maximal in such an extension. Now, $\mathfrak{q}B_\mathfrak{p} \subseteq \mathfrak{q}'B_\mathfrak{p}$ are maximal ideals of $B_\mathfrak{p}$, so they must coincide. □

Proposition (lying over). Let $A \subseteq B$ be an integral extension of rings, and let $\mathfrak{p} \in \operatorname{Spec} A$. Then there is a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. In other words, $\iota^\star : \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

Proof. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\beta"] \\ A_{\mathfrak{p}} \arrow[r] & B_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1}B \end{tikzcd}$$

Let $\mathfrak{m}$ be a maximal ideal of $B_{\mathfrak{p}}$. Then $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is an integral extension, so $\mathfrak{m}$ contracts to a maximal ideal $\mathfrak{m} \cap A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$. But there is exactly one maximal ideal in $A_{\mathfrak{p}}$, namely $\mathfrak{p}A_{\mathfrak{p}}$. Note that $\mathfrak{p}A_{\mathfrak{p}}$ contracts to $\mathfrak{p}$ under the map $A \to A_{\mathfrak{p}}$.

We have that $\mathfrak{m}$ contracts to $\mathfrak{p}$ under the map $A \to A_{\mathfrak{p}} \to B_{\mathfrak{p}}$, but this is the same as the map $A \to B \to B_{\mathfrak{p}}$, so $\beta^{-1}(\mathfrak{m}) \cap A = \mathfrak{p}$. Note that $\beta^{-1}(\mathfrak{m})$ is a prime ideal, as required. □

Theorem (going up)**.** Let $A \subseteq B$ be an integral extension of rings. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be prime ideals in A, and let $\mathfrak{q}_1 \in \operatorname{Spec} B$ be a prime ideal such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then there is a prime ideal $\mathfrak{q}_2 \in \operatorname{Spec} B$ such that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$, and $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

$$\begin{tikzcd} \mathfrak{q}_1 \arrow[r, dashed, "\subseteq"] \arrow[d, "\cap A"'] & \mathfrak{q}_2 \arrow[d, dashed, "\cap A"] \\ \mathfrak{p}_1 \arrow[r, "\subseteq"'] & \mathfrak{p}_2 \end{tikzcd}$$

Proof. We have an injection $A/\mathfrak{p}_1 \to B/\mathfrak{q}_1$ given by $a + \mathfrak{p}_1 \mapsto q + \mathfrak{q}_1$. This is an integral extension, so by lying over, there is a prime ideal $\mathfrak{q}_2/\mathfrak{q}_1$ of $B/\mathfrak{q}_1$ that contracts to $\mathfrak{p}_2/\mathfrak{p}_1$ in $A/\mathfrak{p}_1$. We claim that $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$. In the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A/\mathfrak{p}_1 \arrow[r] & B/\mathfrak{q}_1 \end{tikzcd}$$

we obtain contractions of prime ideals

$$\begin{tikzcd} \mathfrak{p}_2 & \mathfrak{q}_2 \\ \mathfrak{p}_2/\mathfrak{p}_1 \arrow[u, mapsto] & \mathfrak{q}_2/\mathfrak{q}_1 \arrow[u, mapsto] \arrow[l, mapsto] \end{tikzcd}$$

hence $\mathfrak{q}_2$ contracts to $\mathfrak{p}_2$, as required. □

Theorem (going down)**.** Let $A \subseteq B$ be an integral extension of integral domains, and suppose that A is integrally closed (in its field of fractions). Let $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ be prime ideals in A, and let $\mathfrak{q}_1 \in \operatorname{Spec} B$ be a prime ideal such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then there is a prime ideal $\mathfrak{q}_2 \in \operatorname{Spec} B$

such that $\mathfrak{q}_1 \supseteq \mathfrak{q}_2$, and $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

$$\begin{tikzcd} \mathfrak{q}_1 \arrow[d, "\cap A"'] & \mathfrak{q}_2 \arrow[l, dashed, "\supseteq"'] \arrow[d, dashed, "\cap A"] \\ \mathfrak{p}_1 & \mathfrak{p}_2 \arrow[l, "\supseteq"] \end{tikzcd}$$

Proof. Consider the map $A \to B \to B_{\mathfrak{q}_1}$. These maps are injective as B is an integral domain, so we can think of these as inclusions of rings. We want to prove that there is a prime ideal $\mathfrak{n} \in \operatorname{Spec} B_{\mathfrak{q}_1}$ such that $\mathfrak{n} \cap A = \mathfrak{p}_2$. This suffices, as $(\mathfrak{n} \cap B) \cap A = \mathfrak{p}_2$ is a contraction of a prime ideal $\mathfrak{q}_2 = \mathfrak{n} \cap B$ of B contained in $\mathfrak{q}_1$ to $\mathfrak{p}_2 \in \operatorname{Spec} A$. In other words, we want to show that $\mathfrak{p}_2$ is a contracted ideal under the map $A \to B_{\mathfrak{q}_1}$. As contracted ideals are contracted from their own extension, it suffices to show that $(\mathfrak{p}_2 B_{\mathfrak{q}_1}) \cap A \subseteq \mathfrak{p}_2$, noting that the converse inclusion always holds.

Note that $\mathfrak{p}_2 B_{\mathfrak{q}_1} = (\mathfrak{p}_2 B) B_{\mathfrak{q}_1}$. Let $\frac{y}{s} \in (\mathfrak{p}_2 B) B_{\mathfrak{q}_1} \cap A$, where $y \in \mathfrak{p}_2 B$ and $s \in B \setminus \mathfrak{q}_1$. As $A \subseteq B$ is an integral extension, the integral closure of $\mathfrak{p}_2$ in B is $\sqrt{\mathfrak{p}_2 B}$. In particular, y is integral over $\mathfrak{p}_2$. Since A is integrally closed and y is integral over $\mathfrak{p}_2$, the minimal polynomial of $y \in FF(B)$ over $FF(A)$ has the form

$$y^r + u_1 y^{r-1} + \dots + u_r y^0 = 0; \quad u_i \in \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$$

We can write $y = {}^{y}\!/_{s} \cdot s$, where $y, s \in FF(B)$ and $\frac{y}{s} \in FF(A)$. Hence,

$$\left(\frac{y}{s} \cdot s\right)^r + u_1\left(\frac{y}{s} \cdot s\right)^{r-1} + \dots + u_r\left(\frac{y}{s} \cdot s\right)^0 = 0$$

Multiplying by $\left(\frac{s}{y}\right)^r$,

$$s^r + \left(\frac{s}{y}\right)^1 u_1 s^{r-1} + \dots + \left(\frac{s}{y}\right)^r u_r s^0 = 0; \quad u_i \in \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$$

This must be the same minimal polynomial of s as an element of $FF(B)$ over $FF(A)$. As $s \in B$, s is integral over A, so the coefficients in this polynomial must lie in A.

$$\left(\frac{s}{y}\right)^1 u_1, \dots, \left(\frac{s}{y}\right)^r u_r \in A$$

Suppose $\frac{y}{s} \notin \mathfrak{p}_2$. Then

$$u_i = \left(\frac{y}{s}\right)^i \cdot \left(\frac{s}{y}\right)^i u_i$$

But

$$u_1 \in \mathfrak{p}_2; \quad \left(\frac{y}{s}\right)^i \in A \setminus \mathfrak{p}_2; \quad \left(\frac{s}{y}\right)^i u_i \in A$$

By primality, $\left(\frac{s}{y}\right)^i u_i \in \mathfrak{p}_2$. As this holds for all i, the coefficients in the equation for s lie in $\mathfrak{p}_2$, so

$$s^r \in \mathfrak{p}_2 B \subseteq \mathfrak{p}_1 B = (\mathfrak{q}_1 \cap A)B \subseteq \mathfrak{q}_1$$

Hence $s \in \mathfrak{q}_1$ by primality, giving a contradiction. $\square$

5 Primary decomposition

Definition. Let I be an ideal of R. I is
(i) *prime* if $R/I \neq 0$ and 0 is the only zero divisor of R/I;
(ii) *radical* if the only nilpotent element of R/I is zero;
(iii) *primary* if $R/I \neq 0$ and every zero divisor in R/I is nilpotent.

The prime ideals precisely those ideals that are both radical and primary. R is radical but not prime or primary.

Example. (i) Let $R = \mathbb{Z}$. The ideal (6) is radical but not primary, as $R/(6)$ contains zero divisors $2, 3$ which are not nilpotent. The ideal (9) is primary but not radical.

(ii) More generally, let $R = \mathbb{Z}$ and $x \neq 0$. Then (x) is prime if and only if $x = 0$ or $|x|$ is prime, and (x) is radical if and only if x is squarefree. (x) is primary if and only if $x = p^n$ for some prime p and $n \geq 1$.

Proposition. Let I be a proper ideal in R. Then
(i) If I is primary, then $\mathfrak{p} = \sqrt{I}$ is prime. We say I is $\mathfrak{p}$-*primary*.
(ii) If $\sqrt{I}$ is maximal, then I is primary.
(iii) If $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ are $\mathfrak{p}$-primary, then $\bigcap_{i=1}^n \mathfrak{q}_i$ is also $\mathfrak{p}$-primary.
(iv) If I has a *primary decomposition* $I = \bigcap_{i=1}^n \mathfrak{q}_i$ where the $\mathfrak{q}_i$ are primary, then I has a minimal primary decomposition $\bigcap_{j=1}^m \mathfrak{r}_j$ where the $\sqrt{\mathfrak{r}_j}$ are distinct and no $\mathfrak{r}_j$ can be dropped.
(v) If R is Noetherian, then every proper ideal has a primary decomposition.

In $\mathbb{Z}$,

$$(90) = (2) \cap (3^2) \cap (5)$$

Primary decomposition therefore generalises prime factorisation. Note that for a prime ideal $\mathfrak{p}$, if $\mathfrak{p}^n$ is primary, then $\mathfrak{p}^n$ is $\mathfrak{p}$-primary, because $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$.

Example. (i) Not every primary ideal is a power of a prime ideal. For instance, consider $R = k[X, Y]$ and $\mathfrak{q} = (X, Y^2)$. We claim that this is primary. For instance, $\sqrt{\mathfrak{q}} = (X, Y)$ is maximal, so $\mathfrak{q}$ is (X, Y)-primary. Alternatively,

$$k[X, Y]/(X, Y^2) \simeq k[Y]/(Y^2)$$

If $f \in k[Y]$ satisfies $f \in (Y^2)$ so it is a zero divisor, then $Y \mid f$, so $f + (Y^2)$ is nilpotent. Now, if $\mathfrak{q} = \mathfrak{p}^n$, then

$$(X, Y) = \sqrt{\mathfrak{q}} = \sqrt{\mathfrak{p}^n} = \mathfrak{p}$$

But

$$(X, Y) \supsetneq (X, Y^2) \supsetneq (X, Y)^2$$

So $\mathfrak{q}$ is not a power of $\mathfrak{p} = (X, Y)$.

(ii) If $\mathfrak{p}$ is prime, $\mathfrak{p}^n$ need not be primary. Let

$$R = k[X, Y, Z]/(XY - Z^2) = k[\overline{X}, \overline{Y}, \overline{Z}]; \quad \mathfrak{p} = (\overline{X}, \overline{Z})$$

where $\overline{X}, \overline{Y}, \overline{Z}$ are the images of X, Y, Z under the quotient map. We claim that $\mathfrak{p}$ is prime, but $\mathfrak{p}^2$ is not primary. Indeed,

$$R/\mathfrak{p} \simeq k[X,Y,Z]/(X,Z,XY-Z^2) \simeq k[X,Y,Z]/(X,Z) \simeq k[Y]$$

which is an integral domain, so $\mathfrak{p}$ is prime. For the second part,

$$\mathfrak{p}^2 = (\overline{X}^2, \overline{X} \cdot \overline{Z}, \overline{Z}^2)$$

Then $\overline{X} \cdot \overline{Y} = \overline{Z}^2 \in \mathfrak{p}^2$, that is,

$$(\overline{X} + \mathfrak{p}^2)(\overline{Y} + \mathfrak{p}^2) = 0 + \mathfrak{p}^2$$

But $\overline{X} + \mathfrak{p}^2 \neq 0$ and $\overline{Y} + \mathfrak{p}^2 \neq 0$. Hence $\overline{Y} + \mathfrak{p}^2$ is a zero divisor in $R/\mathfrak{p}^2$. Note that

$$R/\mathfrak{p}^2 \simeq k[X,Y,Z]/(XY-Z^2, X^2, XZ, Z^2) \simeq k[X,Y,Z]/(XY, X^2, Z^2)$$

so $Y + \mathfrak{p}^2$ is not nilpotent.

Theorem. Let $\bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition for an ideal I of R, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for each i. Then

(i) (*associated* prime ideals of I) The prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are determined only by I, even though there may not be a unique minimal primary decomposition.

(ii) (*isolated* prime ideals of I) The minimal elements of $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, ordered by inclusion, are exactly the minimal prime ideals of R that contain I. An associated prime ideal that is not isolated is called *embedded*.

(iii) (isolated primary components of I) If $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ are the isolated prime ideals of I for $t \leq n$, then $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ are determined only by I.

Example. Let $R = k[X,Y]$ and $I = (X^2, XY)$. We have primary decompositions

$$I = (X) \cap (X,Y)^2 = (X) \cap (X^2, Y)$$

Note that

$$\sqrt{(X)} = (X); \quad \sqrt{(X,Y)^2} = (X,Y); \quad \sqrt{(X^2,Y)} = (X,Y)$$

The associated primes of I are (X) and (X,Y). The isolated prime is (X) and the embedded prime is (X,Y).

Remark. Let $I = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition with $\sqrt{q_i} = \mathfrak{p}_i$. Suppose $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ are the isolated primes. Then

$$\sqrt{I} = \sqrt{\bigcap_{i=1}^n \mathfrak{q}_i} = \bigcap_{i=1}^n \sqrt{\mathfrak{q}_i} = \bigcap_{i=1}^n \mathfrak{p}_i = \bigcap_{i=1}^t \mathfrak{p}_i$$

This is a primary decomposition of $\sqrt{I}$, and one can check that this is minimal. All associated primes in this decomposition are isolated. Going from I to $\sqrt{I}$, we only 'remember' the isolated primes.

Analogously, let $R = k[T_1, \dots, T_n]$, where $k \subseteq \mathbb{C}$. Then $\mathbb{V}(I) = \mathbb{V}\left(\sqrt{I}\right)$ and $I(\mathbb{V}(I)) = \sqrt{I}$. Hence, taking the algebraic set of I 'remembers' the radical of I and nothing else.

6 Direct and inverse limits

6.1 Limits and completions

Definition. Let $\mathcal{C}$ be a category.

(i) A *directed set* $(I, \leq)$ is a partially ordered set such that for all $a, b \in I$, there exists $c \in I$ such that $a, b \leq c$.

(ii) A *direct system* on a directed set $(I, \leq)$ is a pair $((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ where $X_i \in \operatorname{ob} \mathcal{C}$ and $f_{ij} : X_i \to X_j$, such that $f_{ii} = 1_{X_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$.

(iii) An *inverse system* on $(I, \leq)$ is a pair $((Y_i)_{i \in I}, (h_{ij})_{i \leq j})$ where $Y_i \in \operatorname{ob} \mathcal{C}$ and $h_{ij} : Y_j \to Y_i$, such that $h_{ii} = 1_{X_i}$ and $h_{ik} = h_{ij} \circ h_{jk}$.

Remark. An inverse system in $\mathcal{C}$ is the same as a direct system in $\mathcal{C}^{\mathrm{op}}$.

Example. Let $I = (\mathbb{N}, \leq)$.

(i) Let p be a prime, and let $X_i = \mathbb{F}_{p^{i!}}$. Recall that if $a \mid b$, then there is an embedding $\varphi : \mathbb{F}_{p^a} \to \mathbb{F}_{p^b}$. The collection of embeddings $\mathbb{F}_{p^a} \to \mathbb{F}_{p^b}$ is then given by $x \mapsto (\varphi(x))^{p^c}$ where $0 \leq c < a - 1$. The map $f_{i(i+1)} : \mathbb{F}_{p^{i!}} \to \mathbb{F}_{p^{(i+1)!}}$ is defined to be one such embedding. A general embedding f_{ij} is given by the composite $f_{(j-1)j} \circ \cdots \circ f_{i(i+1)}$. This creates a direct system on I.

(ii) Let $Y_i = \mathbb{Z}/p^i\mathbb{Z}$, and let $h_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ be the natural projection. This is an inverse system on I.

Definition. Let $(I, \leq)$ be a directed set.

(i) Let $D = ((X_i)_{i \in I}, (f_{ij})_{i \leq j})$ be a direct system on I. Then the *direct limit* of D is

$$\varinjlim X_i = \left(\coprod_{i \in I} X_i \right) \Big/ \sim$$

where for $x_i \in X_i$ and $x_j \in X_j$,

$$x_i \sim x_j \iff \exists k \geq i, j,\ f_{ik}(x_i) = f_{jk}(x_j)$$

Equivalently, one can define $\sim$ to be the smallest equivalence relation containing $x_i \sim f_{ij}(x_i)$.

(ii) Let $E = ((Y_i)_{i \in I}, (h_{ij})_{i \leq j})$ be an inverse system on I. Then the *inverse limit* of E is

$$\varprojlim Y_i = \left\{ \mathbf{y} \in \prod_{X_i} \,\middle|\, \forall i \leq j,\ y_i = h_{ij}(y_j) \right\}$$

Example. (i) $\mathbb{F}_p^{\mathrm{alg}} = \varinjlim \mathbb{F}_{p^{i!}}$ is an algebraic closure of $\mathbb{F}_p$. First, $\mathbb{F}_p^{\mathrm{alg}}$ is algebraic over $\mathbb{F}_p$. Indeed, for $[x] \in \mathbb{F}_p^{\mathrm{alg}}$, we have $x \in \mathbb{F}_p^{i!}$ for some $i \geq 1$. Then $x^{p^{i!}} - x = 0$. Hence

$$[x]^{p^{i!}} - [x] = [x^{p^{i!}} - x] = [0]$$

Further, $\mathbb{F}_p^{\mathrm{alg}}$ is algebraically closed. Any polynomial $h \in \mathbb{F}_p^{\mathrm{alg}}[T]$ has coefficients in $\mathbb{F}_p^{\mathrm{alg}}$, so in particular h arises from an element of $\mathbb{F}_{p^{i!}}[T]$ for some i. This element splits under some

$\mathbb{F}_{p^{i!}} \to \mathbb{F}_{p^e}$, so it splits under some $\mathbb{F}_{p^{i!}} \to \mathbb{F}_{p^{e!}}$. Hence it splits under $h_{ij} : \mathbb{F}_{p^{i!}} \to \mathbb{F}_{p^{j!}}$, so h splits in the direct limit $\mathbb{F}_p^{\mathrm{alg}}$.

(ii) Define $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$. This is the ring of *p-adic integers*. For example, writing numbers in base $p = 5$,

$$1 = (1 + 5^1\mathbb{Z}, 1 + 5^2\mathbb{Z}, 1 + 5^3\mathbb{Z}, \dots)$$
$$-1 = (4 + 5^1\mathbb{Z}, 44 + 5^2\mathbb{Z}, 444 + 5^3\mathbb{Z}, \dots)$$

In every position in such an expansion, we 'expose' another digit of the p-adic integer to the left.

Definition. Let R be a ring, and let $\mathfrak{a}$ be an ideal of R. Then the $\mathfrak{a}$*-adic completion* of R is

$$\hat{R} = \varprojlim R/\mathfrak{a}^i$$

where the inverse limit is taken over the directed system $(\mathbb{N}, \leq)$ with morphisms given by the natural projections.

Example. (i) If $R = \mathbb{Z}$ and $\mathfrak{a} = (p)$, then $\hat{R} = \mathbb{Z}_p$.

(ii) If $R = k[T]$ and $\mathfrak{a} = (T)$, then

$$\hat{R} = \varprojlim k[T]/(T^i) = k[\![t]\!]$$

Definition. Let M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Then the $\mathfrak{a}$*-adic completion* of M is

$$\hat{M} = \varprojlim M/\mathfrak{a}^i M$$

which is naturally an $\hat{R}$-module.

We can make the following more general definition.

Definition. Let M be an R-module.

(i) A *filtration* of M is a sequence $(M_n)_{n\geq 1}$ of submodules of M such that $M_0 = M$ and $M_n \supseteq M_{n+1}$ for each n.

(ii) The *completion* of M with respect to a filtration $(M_n)_{n\geq 1}$ is $\varprojlim M/M_n$.

Theorem. Let R be a Noetherian ring, and let $\mathfrak{a}$ be an ideal of R. Then,

(i) the $\mathfrak{a}$-adic completion $\hat{R}$ is Noetherian;

(ii) the functor $\hat{R} \otimes_R (-)$ is exact;

(iii) if M is a finitely generated R-module, then the natural map $\hat{R} \otimes_R M \to \hat{M}$ is an $\hat{R}$-linear isomorphism.

Thus $\mathfrak{a}$-adic completion is an exact functor from the category of finitely generated R-modules if R is Noetherian.

6.2 Graded rings and modules

Definition. A *graded ring* is a ring $A = \bigoplus_{n=0}^{\infty} A_n$, where each A_n is an additive subgroup of A, such that $A_m A_n \subseteq A_{m+n}$.

Proposition. A_0 is a subring of A.

Proof. It is clearly a subgroup closed under multiplication, so it suffices to check that it contains the identity element of A. We have

$$1_A = \sum_{i=0}^{m} y_i; \quad y_i \in A_i$$

For $z_n \in A_n$,

$$z_n = \sum_{i=0}^{m} y_i z_n$$

z_n is an element of A_n, and each term $y_i z_n$ is an element of A_{n+i}. But since the sum is direct, we must have $z_n = y_0 z_n$, so $z = y_0 z$ for all $z \in A$. Hence $y_0 \in A_0$ is the identity element. □

Remark. Each A_n is an A_0-module as $A_0 A_n \subseteq A_n$.

Example. The polynomial ring in finitely many variables has a grading: $k[T_1, \dots, T_m] = \bigoplus_{n=0}^{\infty} A_n$ where A_n is the set of homogeneous polynomials of degree n.

Definition. Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded ring. A *graded A-module* is an A-module $M = \bigoplus_{n=0}^{\infty} M_n$ such that $A_m M_n \subseteq M_{m+n}$.

For a graded ring A, we define $A_+ = \bigoplus_{n=1}^{\infty} A_n = \ker(A \twoheadrightarrow A_0)$. This is an ideal of A, and ${}^{A}\!/\!{}_{A_+} \simeq A_0$.

Proposition. Let $A = \bigoplus_{i=0}^{\infty} A_n$ be a graded ring. Then the following are equivalent:
(i) A is Noetherian;
(ii) A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof. Hilbert's basis theorem shows that (ii) implies (i). For the converse, A_0 is Noetherian as it is isomorphic to a quotient of the Noetherian ring A. Note that A_+ is generated by the set of homogeneous elements of positive degree. By (i), A_+ an ideal in a Noetherian ring so is generated by a finite set $\{x_1, \dots, x_s\}$, and we can take each x_i to be homogeneous, say, $x_i \in A_{k_i}$ where $k_i > 0$. Let A' be the A_0-subalgebra of A generated by $\{x_1, \dots, x_s\}$; we want to show $A' = A$. It suffices to show that $A_n \subseteq A'$ for every $n \geq 0$, which we will show by induction. The case $n = 0$ is clear.

Let $n > 0$, and let $y \in A_n$. Note that $y \in A_+$, so

$$y = \sum_{i=1}^{s} r_i x_i$$

where $r_i \in A$ and $x_i \in A_{k_i}$. Applying the projection to A_n,

$$y = \sum_{i=1}^{s} a_i x_i; \quad a_i \in A_{n-k_i}$$

where a_i is the $(n - k_i)$ homogeneous part of r_i. As k_i is positive, the inductive hypothesis implies that each a_i can be written as a polynomial in $x_1, \dots, x_s$ with coefficients in A_0, giving $y \in A'$ as required. □

Definition. Let $\mathfrak{a}$ be an ideal of R, and let M be an R-module. Then a filtration $(M_n)_{n \geq 0}$ is an $\mathfrak{a}$-*filtration* if $\mathfrak{a}M_n \subseteq M_{n+1}$ for each $n \geq 0$. An $\mathfrak{a}$-filtration $(M_n)_{n \geq 0}$ is *stable* if there exists $n_0 \geq 0$ such that $\mathfrak{a}M_n = M_{n+1}$ for all $n \geq n_0$.

Example. $(\mathfrak{a}^n M)_{n \geq 0}$ is a stable $\mathfrak{a}$-filtration of M.

Definition. Let $\mathfrak{a}$ be an ideal in R. The *associated graded ring* is

$$G_{\mathfrak{a}}(R) = \bigoplus_{n \geq 0} {}^{\mathfrak{a}^n}\!/\!_{\mathfrak{a}^{n+1}}; \quad \mathfrak{a}^0 = R$$

This is a ring by defining

$$(x + \mathfrak{a}^{n+1})(y + \mathfrak{a}^{m+1}) = xy + \mathfrak{a}^{n+m+1}; \quad x \in \mathfrak{a}^n, y \in \mathfrak{a}^m$$

Definition. Let M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Let $(M_n)_{n \geq 0}$ be an $\mathfrak{a}$-filtration of M. The *associated graded module* is

$$G(M) = \bigoplus_{n \geq 0} {}^{M_n}\!/\!_{M_{n+1}}$$

This is a module over $G_{\mathfrak{a}}(R)$ by defining

$$(x + \mathfrak{a}^{n+1})(m + M_{\ell+1}) = xm + M_{n+\ell+1}$$

Proposition. Let R be a Noetherian ring, and let $\mathfrak{a}$ be an ideal of R. Then
(i) the associated graded ring $G_{\mathfrak{a}}(R)$ is Noetherian; and
(ii) if M is a finitely generated R-module and $(M_n)_{n \geq 0}$ is a stable $\mathfrak{a}$-filtration of M, then the associated graded module $G(M)$ is a finitely generated $G_{\mathfrak{a}}(R)$-module.

Proof. Part (i). Let R be Noetherian. Then let $\mathfrak{a} = (x_1, \dots, x_s)$, and write $\overline{x}_i$ for the image of x_i in ${}^{\mathfrak{a}}\!/\!_{\mathfrak{a}^2}$. Note that

$$G_{\mathfrak{a}}(R) = {}^{R}\!/\!_{\mathfrak{a}} \oplus {}^{\mathfrak{a}}\!/\!_{\mathfrak{a}^2} \oplus {}^{\mathfrak{a}^2}\!/\!_{\mathfrak{a}^3} \oplus \cdots$$

$G_{\mathfrak{a}}(R)$ is generated as an ${}^{R}\!/\!_{\mathfrak{a}}$-algebra by $\overline{x}_1, \dots, \overline{x}_s$, by taking sums and products. Note that ${}^{R}\!/\!_{\mathfrak{a}}$ is Noetherian, so $G_{\mathfrak{a}}(R)$ is Noetherian by Hilbert's basis theorem.

Part (ii). Let $(M_n)_{n\geq 0}$ be a stable $\mathfrak{a}$-filtration of M. Then there exists n_0 such that for all $n \geq n_0$, we have $M_{n_0+r} = \mathfrak{a}^r M_{n_0}$. Thus $G(M)$ is generated as a $G_\mathfrak{a}(R)$-module by

$$^{M_0}\!/_{M_1} \oplus {}^{M_1}\!/_{M_2} \oplus \cdots \oplus {}^{M_{n_0}}\!/_{M_{n_0+1}}$$

Each factor $^{M_i}\!/_{M_{i+1}}$ is a Noetherian R-module, as they are quotients of Noetherian modules, and are annihilated by $\mathfrak{a}$. In particular, $G(M)$ is a finitely generated $G_\mathfrak{a}(R)$-module, say by $x_1, \dots, x_s$. □

Definition. Let M be an R-module. We say that filtrations $(M_n), (M'_n)$ of M are *equivalent* if there exists n_0 such that for all $n \geq 0$, we have $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$.

Lemma. Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module, and let $(M_n)_{n\geq 0}$ be a stable $\mathfrak{a}$-filtration of M. Then $(M_n)_{n\geq 0}$ is equivalent to $(\mathfrak{a}^n M)_{n\geq 0}$.

Proof. As $(M_n)_{n\geq 0}$ is an $\mathfrak{a}$-filtration, for all $n \geq 0$, we have

$$M_n \supseteq \mathfrak{a}M_{n-1} \supseteq \mathfrak{a}^2 M_{n-2} \supseteq \cdots \supseteq \mathfrak{a}^n M \supseteq \mathfrak{a}^{n+n_0} M$$

For the other direction, as the filtration is stable, there exists n_0 such that for each $n \geq n_0$, we have $\mathfrak{a}M_n = M_{n+1}$. Then $M_{m+n_0} = \mathfrak{a}^n M_{n_0} \subseteq \mathfrak{a}^n M$ as required. □

6.3 Artin–Rees lemma

Definition. Let $\mathfrak{a}$ be an ideal of R. Let M be an R-module, and let $(M_n)_{n\geq 0}$ be an $\mathfrak{a}$-filtration of M. Then we define

$$R^\star = \bigoplus_{n\geq 0} \mathfrak{a}^n; \quad M^\star = \bigoplus_{n\geq 0} M_n$$

Note that $R^\star$ is a graded ring, as for $x \in \mathfrak{a}^n, y \in \mathfrak{a}^\ell$, we have $xy \in \mathfrak{a}^{n+\ell}$. As $(M_n)_{n\geq 0}$ is an $\mathfrak{a}$-filtration, $M^\star$ is a graded $R^\star$-module. Indeed, for $x \in \mathfrak{a}^n$ and $m \in M_\ell$, we have $xm \in M_{n+\ell}$ as required.

If R is Noetherian, the ideal $\mathfrak{a}$ is finitely generated, say by $x_1, \dots, x_r$. Then $R^\star$ is generated as an R-algebra by $x_1, \dots, x_r$ by taking sums and products. By Hilbert's basis theorem, $R^\star$ is a Noetherian ring.

Lemma. Let R be a Noetherian ring, and let $\mathfrak{a}$ be an ideal of R. Let M be a finitely generated R-module, and let $(M_n)_{n\geq 0}$ be an $\mathfrak{a}$-filtration of M. Then, the following are equivalent:
(i) $M^\star$ is finitely generated as an $R^\star$-module;
(ii) the $\mathfrak{a}$-filtration $(M_n)_{n\geq 0}$ is stable.

Proof. First, note that each M_n is a finitely generated R-module. Indeed, R is a Noetherian ring and M is finitely generated, so M is a Noetherian module, or equivalently, every submodule is finitely generated. Now, consider

$$M_n^\star = M_0 \oplus \cdots \oplus M_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots$$

This is an $R^\star$-submodule of $M^\star$. Note that $(M_n^\star)_{n\geq 0}$ is an ascending chain of $R^\star$-submodules of $M^\star$, and this chain stabilises if and only if the $\mathfrak{a}$-filtration $(M_n)_{n\geq 0}$ is stable.

(i) implies (ii). As R is Noetherian, so is $R^\star$ by the discussion above. By assumption, $M^\star$ is finitely generated as a module over a Noetherian ring, so it is Noetherian. Hence the ascending chain $(M_n^\star)_{n\geq 0}$ stabilises, giving the result.

(ii) implies (i). Suppose $(M_n)_{n\geq 0}$ is stable. Then $(M_n^\star)_{n\geq 0}$ stabilises at some $n_0 \geq 0$, so

$$M^\star = \bigcup_{n\geq 0} M_n^\star = M_{n_0}^\star$$

Now, note that $M_0 \oplus \cdots \oplus M_{n_0}$ generatees $M_{n_0}^\star$ as an $R^\star$-module. Each M_n is a finitely generated R-module, so $M_0 \oplus \cdots \oplus M_{n_0}$ is also finitely generated as an R-module. So these generators span $M_{n_0}^\star = M^\star$ as an $R^\star$-module, as required. □

Proposition (Artin–Rees)**.** Let R be a Noetherian ring, and let $\mathfrak{a}$ be an ideal of R. Let M be a finitely generated R-module, and let $(M_n)_{n\geq 0}$ be a stable $\mathfrak{a}$-filtration of M. Then for any submodule $N \leq M$, $(N \cap M_n)_{n\geq 0}$ is a stable $\mathfrak{a}$-filtration of N.

Thus, stable filtrations pass to submodules.

Proof. First, we show that $(N \cap M_n)_{n\geq 0}$ is indeed an $\mathfrak{a}$-filtration.

$$\mathfrak{a}(N \cap M_n) \subseteq N \cap \mathfrak{a}M_n \subseteq N \cap M_{n+1}$$

Now, define

$$M^\star = \bigoplus_{n\geq 0} M_n; \quad N^\star = \bigoplus_{n\geq 0} (N \cap M_n)$$

Note that $M^\star$ is an $R^\star$-submodule of $N^\star$. As R is Noetherian, so is $R^\star$. Then as $(M_n)_{n\geq 0}$ is stable, $M^\star$ is a finitely generated $R^\star$-module by the previous lemma. Thus $M^\star$ is a Noetherian $R^\star$-module. Its submodule $N^\star$ is then finitely generated, so $(N \cap M_n)_{n\geq 0}$ is stable. □

7 Dimension theory

7.1 ???

Definition. Let $\mathfrak{p}$ be a prime ideal of R. The *height* of $\mathfrak{p}$, denoted $\mathrm{ht}(p)$, is

$$\mathrm{ht}(\mathfrak{p}) = \sup\{d \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{p}; \mathfrak{p}_i \in \operatorname{Spec} R\}$$

The *(Krull) dimension* of R is

$$\dim R = \sup\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} R\} = \sup\{\mathrm{ht}(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{mSpec} R\}$$

Remark. The height of a prime ideal $\mathfrak{p}$ is the Krull dimension of the localisation $R_\mathfrak{p}$. In particular,

$$\dim R = \sup\left\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\right\} = \sup\{\dim R_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{mSpec} R\}$$

So the problem of computing dimension can be reduced to computing dimension of local rings.

Definition. Let I be a proper ideal of R. Then the *height* of I is

$$\mathrm{ht}(I) = \inf\{\mathrm{ht}(\mathfrak{p}) \mid I \subseteq \mathfrak{p}\}$$

Proposition. Let $A \subseteq B$ be an integral extension of rings. Then,
(i) $\dim A = \dim B$; and
(ii) if A, B are integral domains and k-algebras for some field k, they have the same transcendence degree over k.

We prove part (i); the second part is not particularly relevant for this course.

Proof. First, we show that $\dim A \leq \dim B$. Consider a chain of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ in $\operatorname{Spec} A$. By the lying over theirem and the going up theorem, we obtain a chain of prime ideals $\mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_d$ in $\operatorname{Spec} B$. As $\mathfrak{p}_i = \mathfrak{q}_i \cap A$ and $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$, we must have $\mathfrak{q}_i \neq \mathfrak{q}_{i+1}$. So this produces a chain of length d in B, as required.

Now consider a chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ in $\operatorname{Spec} B$. Contracting each ideal, we produce a chain $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_d$ in $\operatorname{Spec} A$. Suppose that $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ contract to the same prime ideal $\mathfrak{p}_i$ in $\operatorname{Spec} A$. Note that $\mathfrak{q}_i \subseteq \mathfrak{q}_{i+1}$, so by incomparability, they must be equal, but this is a contradiction. □

Remark. If A is a finitely generated k-algebra for some field k, then by Noether normalisation, we obtain a k-algebra embedding $k[T_1, \dots, T_d] \to A$, and the extension is integral. Thus $\dim A = \dim k[T_1, \dots, T_d]$. One can show that $\dim k[T_1, \dots, T_d] = d$, and hence that the integer d obtained by Noether normalisation is uniquely determined by A and k.

7.2 Hilbert polynomials

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring, so A_0 is Noetherian and A is finitely generated as an A_0-algebra. Now let $M = \bigoplus_{n \geq 0} M_n$ be a finitely generated graded A-module. Then each M_n is an A_0-module.

We claim that M_n is finitely generated as an A_0-module. Indeed, $M = \operatorname{span}_A \{m_1, \dots, m_t\}$, and the m_i can be taken to be homogeneous, say, $m_i \in M_{r_i}$. Then

$$M_n = \{a_1 m_1 + \dots + a_t m_t \mid a_i \in A_{n-r_i}\}$$

Let $x_1, \dots, x_s$ generate A as an A_0-algebra, where $x_i \in A_{k_i}, k_i > 0$. Then

$$M_n = \operatorname{span}_{A_0} \left\{ x_1^{e_1} \dots x_t^{e_t} m_i \,\middle|\, 1 \leq i \leq t, e_i \geq 0, \sum_{i=1}^{s} k_i e_i = n - r_i \right\}$$

and the right-hand side is a finite set.

We will make the further assumption that A_0 is Artinian. Hence, each M_n is a finitely generated module over a ring that is both Noetherian and Artinian, so each M_n is Noetherian and Artinian as an A_0-module. Further, each M_n is of finite length $\ell(M_n) < \infty$; it has a *composition series* of finite length. Note that if $A_0 = k$ is a field, then $\ell(M_n) = \dim_k M_n$.

Definition. Let A, M be as above. Then the *Poincaré series* of M is

$$P(M,T) = \sum_{n=0}^{\infty} \ell(M_n) T^n \in \mathbb{Z}[\![T]\!]$$

Theorem (Hilbert–Serre theorem)**.** Let A be generated by $x_1, \dots, x_s$ as an A_0-module with $x_i \in A_{k_i}$ for $k_i > 0$. The Poincaré series $P(M,T)$ is a rational function of the form

$$\frac{f(T)}{\prod_{i=1}^{s}(1 - T^{k_i})}; \quad f \in \mathbb{Z}[T]$$

Proof. For the base case $s = 0$, we must have $A = A_0$, so M is a finitely generated A_0-module, say, $M = \operatorname{span}_{A_0} S$ where S is a finite subset of $M_0 \oplus \dots \oplus M_n$. Thus there exists n_0 such that $M_m = 0$ for all $m > n_0$. In particular, $P(M,T)$ is a polynomial.

For the inductive step, let

$$M = \bigoplus_{n \in \mathbb{Z}} M_n; \quad M_\ell = 0 \text{ if } \ell < 0$$

Let $f : M_n \to M_{n+k_s}$ be the homomorphism given by multiplication by x_s. We obtain the exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{f} M_{n+k_s} \longrightarrow L_{n+k_s} \longrightarrow 0$$

where $K_n = \ker f$ and $L_{n+k_s} = \operatorname{coker} f$. Then let $K = \bigoplus_{n\in\mathbb{Z}} K_n$ and $L = \bigoplus_{n\in\mathbb{Z}} L_n$. These are graded A-modules, and K is a submodule of M. Note that K and L are annihilated by x_s. Applying the length function to the exact sequence, we obtain

$$\ell(K_n) - \ell(M_n) + \ell(M_{n+k_s}) - \ell(L_{n+k_s}) = 0$$

Multiplying by T^{n+k_s},

$$\ell(M_{n+k_s})T^{n+k_s} - T^{k_s}\ell(M_n)T^n = \ell(L_{n+k_s})T^{n+k_s} - T^{k_s}\ell(K_n)T^n$$

Then, taking the sum over all integers,

$$P(M,T) - T^{k_s}P(M,T) = (1 - T^{k_s})P(M,T) = P(L,T) - T^{k_s}P(K,T)$$

By the inductive hypothesis,

$$(1 - T^{k_s})P(M,T) = \frac{f_1(T)}{\prod_{i=1}^{s-1}(1 - T^{k_s})} + \frac{f_2(T)}{\prod_{i=1}^{s-1}(1 - T^{k_s})}$$

as required. □

In particular, this rational function is holomorphic almost everywhere, with potentially a pole of some order at 1. Let $d(M)$ be the order of the pole of $P(M,T)$ at $T = 1$. One can show that if $M \neq 0$, then $d(M) \geq 0$.

Example. Let $A = k[T_1, \dots, T_s] = \bigoplus_{n\geq 0} A_n$ where A_n is the set of homogeneous polynomials of degree n. Then A is generated as an $A_0 = k$-algebra by $\{T_1, \dots, T_s\}$. For this choice of generators, $k_1 = \dots = k_s = 1$. The length of A_n is $\dim_k A_n = \binom{n+s-1}{n}$, which is a polynomial of degree $s-1$ in n over $\mathbb{Q}$. The Poincaré series of A over itself is

$$P(A,T) = \sum_{n\geq 0} \binom{n+s-1}{n} T^n = \frac{1}{(1-T)^s}$$

Proposition. If $k_1 = \dots = k_s = 1$, then there exists a *Hilbert polynomial* $HP_M \in \mathbb{Q}[T]$ and $n_0 \geq 0$ such that

$$\ell(M_n) = HP_M(n)$$

for all $n \geq n_0$. In addition, $\deg HP_M = d(M) - 1$ where $d(M)$ is the order of the pole of $P(M,T)$ at $T = 1$.

Proof. Let $d = d(M) \geq 0$. Then,

$$P(M,T) = \sum_{n\geq 0} \ell(M_n) T^n = \frac{f(T)}{(1-T)^d}; \quad f \in \mathbb{Z}[T], f(1) \neq 0$$

Let

$$f = \sum_{k=0}^{\deg f} a_k T^k; \quad a_k \in \mathbb{Z}$$

Note that

$$\frac{1}{(1-T)^d} = \sum_{j=0}^{\infty} \underbrace{\binom{j+d-1}{j}}_{b_j} T^j$$

Thus, for $n \geq \deg f$,

$$\ell(M_n) = \sum_{i=0}^{\deg f} a_i b_{n-i}$$

Note that b_j is a polynomial in j over $\mathbb{Q}$ of degree $d-1$ with leading coefficient $\frac{1}{(d-1)!}$. Then $\ell(M_n)$ is a polynomial p in n over $\mathbb{Q}$ for $n \geq \deg f$. Then $\deg p \leq d-1$, and the coefficient of T^{d-1} in p is

$$\sum_{i=0}^{\deg f} a_i \cdot \frac{1}{(d-1)!} = \frac{f(1)}{(d-1)!} \neq 0$$

so the degree is exactly $d-1$. □

7.3 Dimension theory of local Noetherian rings

Lemma. Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then

(i) an ideal $\mathfrak{q}$ of A is $\mathfrak{m}$-primary if and only if there exists $t \geq 1$ such that $\mathfrak{m}^t \subseteq \mathfrak{q} \subseteq \mathfrak{m}$;

(ii) if $\mathfrak{q}$ is $\mathfrak{m}$-primary, then ${}^{A}\!/_{\mathfrak{q}}$ is Artinian.

Proof. Part (i). Given an ideal $\mathfrak{q}$ between $\mathfrak{m}^t$ and $\mathfrak{m}$, taking radicals we obtain

$$\sqrt{\mathfrak{m}^t} \subseteq \sqrt{\mathfrak{q}} \subseteq \sqrt{\mathfrak{m}}$$

Hence $\sqrt{\mathfrak{q}} = \mathfrak{m}$ and thus $\mathfrak{q}$ is $\mathfrak{m}$-primary. Conversely, if $\mathfrak{q}$ is $\mathfrak{m}$-primary, $\left(\sqrt{\mathfrak{q}}\right)^t \subseteq \mathfrak{q}$ for some t as A is Noetherian, so $\mathfrak{m}^t \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ as required.

Part (ii). $\left({}^{A}\!/\!_{\mathfrak{q}}, {}^{\mathfrak{m}}\!/\!_{\mathfrak{q}}\right)$ is a Noetherian local ring. If $\mathfrak{q} \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, then taking radicals,

$$\mathfrak{m} = \sqrt{\mathfrak{q}} \subseteq \mathfrak{p} \subseteq \mathfrak{m}$$

Hence $\mathfrak{p} = \mathfrak{m}$. In particular, the spectrum of ${}^{A}\!/\!_{\mathfrak{q}}$ is the single ideal ${}^{\mathfrak{m}}\!/\!_{\mathfrak{q}}$. Thus its dimension is zero, and so the quotient is Artinian. □

Theorem (dimension theorem)**.** If A is a Noetherian local ring, then

$$\dim A = \delta(A) = d(G_{\mathfrak{m}}(A))$$

where $\delta(A) = \min\{\delta(\mathfrak{q}) \mid \mathfrak{q} \subseteq A \text{ is } \mathfrak{m}\text{-primary}\}$ and $\delta(\mathfrak{q})$ is the minimal number of generators of $\mathfrak{q}$, and where the right-hand side is the order of the pole at $T = 1$ of the rational function equal to the Poincaré series

$$\sum_{n \geq 0} \ell\left({}^{\mathfrak{m}^n}\!/\!_{\mathfrak{m}^{n+1}}\right) T^n$$

of the associated graded ring.

Proof. We will show that $\delta \geq d \geq \dim \geq \delta$.

Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of A, generated by $x_1, \dots, x_s$ where $s = \delta(\mathfrak{q})$. Then

$$G_{\mathfrak{q}}(A) = {}^{A}\!/\!_{\mathfrak{q}} \oplus {}^{\mathfrak{q}}\!/\!_{\mathfrak{q}^2} \oplus \bigoplus\nolimits_{n \geq 2} {}^{\mathfrak{q}^n}\!/\!_{\mathfrak{q}^{n+1}}$$

The first factor ${}^{A}\!/\!_{\mathfrak{q}}$ is Artinian, and the images of $x_1, \dots, x_s$ generate $G_{\mathfrak{q}}(A)$ as an ${}^{A}\!/\!_{\mathfrak{q}}$-algebra, where the x_i are of degree 1. Then $\ell\left({}^{\mathfrak{q}^n}\!/\!_{\mathfrak{q}^{n+1}}\right) < \infty$. From the theorem on Hilbert polynomials, $\ell\left({}^{\mathfrak{q}^n}\!/\!_{\mathfrak{q}^{n+1}}\right)$ is a polynomial in n of degree at most $\delta(\mathfrak{q}) - 1$, for sufficiently large n.

Fix some $\mathfrak{m}$-primary ideal $\mathfrak{q}_0$ such that $\delta(\mathfrak{q}_0) = \delta(A)$. We consider two special cases: $\mathfrak{q} = \mathfrak{q}_0$ and $\mathfrak{q} = \mathfrak{m}$. For $\mathfrak{q}$, we have

$$\deg \ell\left({}^{\mathfrak{q}_0^n}\!/\!_{\mathfrak{q}^0_{n+1}}\right) \leq \delta(A) - 1$$

As

$$\ell\left({}^{A}\!/\!_{\mathfrak{q}_0^n}\right) = \sum_{i=0}^{n-1} \ell\left({}^{\mathfrak{q}_0^i}\!/\!_{\mathfrak{q}_0^{i+1}}\right)$$

we have

$$\deg \ell\left({}^{A}\!/\!_{\mathfrak{q}_0^n}\right) \leq \delta(A)$$

For $\mathfrak{m}$,

$$\deg \ell\left({}^{\mathfrak{m}^n}\!/\!_{\mathfrak{m}^{n+1}}\right) = d(G_{\mathfrak{m}}(A)) - 1$$

and hence

$$\deg \ell\left(A/_{\mathfrak{m}^n}\right) = d(G_{\mathfrak{m}})(A)$$

Now, there exists $t \geq 1$ such that $\mathfrak{m}^t \subseteq \mathfrak{q}_0 \subseteq \mathfrak{m}$. Then

$$\ell\left(A/_{\mathfrak{m}^n}\right) \leq \ell\left(A/_{\mathfrak{q}_0^n}\right) \leq \ell\left(A/_{\mathfrak{m}^{tn}}\right)$$

But all of these terms are eventually polynomial, and the degrees of the left-hand and right-hand sides are the same, so we must have $\ell\left(A/_{\mathfrak{q}_0^n}\right) = \ell\left(A/_{\mathfrak{m}^n}\right)$.

Proposition. $\delta(A) \geq d(G_{\mathfrak{m}})(A)$

Proof.

$$\delta(A) = \delta(\mathfrak{q}_0) \geq \deg \ell\left(A/_{\mathfrak{q}_0^n}\right) = \deg \ell\left(A/_{\mathfrak{m}^n}\right) = d(G_{\mathfrak{m}}(A))$$

□

Proposition. If $x \in \mathfrak{m}$ is not a zero divisor, then

$$d\left(G_{(\mathfrak{m}/_{xA})}\left(A/_{xA}\right)\right) \leq d(G_{\mathfrak{m}}(A)) - 1$$

This proposition allows us to prove results by induction on d.

Proof. We have a local ring $\left(A/_{xA}, \mathfrak{m}/_{xA}\right)$. Then

$$d(G_{\mathfrak{m}}(A)) = \deg \ell\left(A/_{\mathfrak{m}^n}\right)$$

and

$$d\left(G_{(\mathfrak{m}/_{xA})}\left(A/_{xA}\right)\right) = \deg \ell\left(A/xA/_{(\mathfrak{m}/xA)^n}\right) = \deg \ell\left((\mathfrak{m}^n + xA)/_{xA}\right)$$

We want to show that

$$\deg \ell\left((\mathfrak{m}^n + xA)/_{xA}\right) \leq \deg \ell\left(A/_{\mathfrak{m}^n}\right) - 1$$

We have the short exact sequence

$$0 \longrightarrow (\mathfrak{m}^n + xA)/_{\mathfrak{m}^n} \longrightarrow A/_{\mathfrak{m}^n} \longrightarrow A/_{(\mathfrak{m}^n + xA)} \longrightarrow 0$$

By the second isomorphism theorem,

$$(\mathfrak{m}^n + xA)/_{\mathfrak{m}^n} \cong xA/_{(\mathfrak{m}^n \cap xA)}$$

Thus, by additivity of length,

$$\ell\left(A/_{\mathfrak{m}^n + xA}\right) = \ell\left(A/_{\mathfrak{m}^n}\right) - \ell\left(xA/_{(\mathfrak{m}^n \cap xA)}\right)$$

Note that $(\mathfrak{m}^n)_{n\geq 0}$ is a stable $\mathfrak{m}$-filtration of A, so $(\mathfrak{m}^n \cap xA)_{n\geq 0}$ is a stable $\mathfrak{m}$-filtration of the submodule xA by the Artin–Rees lemma. Then $(\mathfrak{m}^n \cap xA)_{n\geq 0}$ is equivalent to the $\mathfrak{m}$-filtration $(\mathfrak{m}^n xA)_{n\geq 0}$. This equivalence implies that there exists n_0 such that

$$\ell\left(xA/_{(\mathfrak{m}^n xA)}\right) \leq \ell\left(xA/_{(\mathfrak{m}^{n+n_0} \cap xA)}\right); \quad \ell\left(xA/_{(\mathfrak{m}^n \cap xA)}\right) \leq \ell\left(xA/_{(\mathfrak{m}^{n+n_0} xA)}\right)$$

Hence the polynomials have the same leading term, and so the degree of $\ell\left(A/_{\mathfrak{m}^n}\right)$ must decrease. □

Proposition. $d(G_\mathfrak{m}(A)) \geq \dim A$.

Proof. We can prove this by induction using the previous proposition. ☐

Proposition. $\dim A \leq \delta(A)$. That is, there exists an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ that is generated by $d = \dim A$ elements.

Proof. As $\mathfrak{m}$ is the unique maximal ideal, we must have $\operatorname{ht}(\mathfrak{m}) = d$. Also, $\operatorname{ht}(\mathfrak{p}) < d$ for any prime $\mathfrak{p} \neq \mathfrak{m}$. We will form an ideal $\mathfrak{q}$ generated by d elements such that $\operatorname{ht}(\mathfrak{q}) \geq d$. This suffices, as then for every minimal prime ideal $\mathfrak{p}$ of $\mathfrak{q}$, we must have $\operatorname{ht}(\mathfrak{p}) = d$ and thus $\mathfrak{p} = \mathfrak{m}$, giving $\sqrt{\mathfrak{q}} = \mathfrak{m}$ so $\mathfrak{p}$ is $\mathfrak{m}$-primary as required.

Construct $x_1, \dots, x_d$ inductively such that $\operatorname{ht}(\mathfrak{q}_i) \geq i$ where $\mathfrak{q}_i = (x_1, \dots, x_i)$. For the base case, we take $\mathfrak{q}_0 = (0)$. For the inductive step, we assume that $\mathfrak{q}_{i-1} = (x_1, \dots, x_{i-1})$ has already been constructed, with $i - 1 < d$ and $\operatorname{ht}(\mathfrak{q}_{i-1}) \geq i - 1$. We claim that there are only finitely many prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ that contain $\mathfrak{q}_{i-1}$ and have height exactly $i-1$. Indeed, $\operatorname{ht}(\mathfrak{q}_{i-1}) \geq i-1$, so each $\mathfrak{p}_j$ is a minimal prime ideal of $\mathfrak{q}_{i-1}$, and in a Noetherian ring, every ideal has only finitely many minimal primes. We know that $i - 1 < d = \operatorname{ht}(\mathfrak{m})$, so $\mathfrak{m} \nsubseteq \mathfrak{p}_j$ for all j. Therefore, $\mathfrak{m} \nsubseteq \bigcup_j \mathfrak{p}_j$ by the prime avoidance lemma. Take $x_i \in \mathfrak{m} \setminus \bigcup_j \mathfrak{p}_j$, and define $\mathfrak{q}_i = (x_1, \dots, x_{i-1}, x_i)$. Now, if $\mathfrak{p}$ is a prime ideal that contains $\mathfrak{q}_i$, as $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$, we must have $\operatorname{ht}(p) \geq i$ as required. ☐

☐

Corollary (Krull's height theorem)**.** Let A be a Noetherian ring, and let $\mathfrak{a} = (x_1, \dots, x_r)$ be an ideal of A. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$. Then $\operatorname{ht}(\mathfrak{p}) \leq r$.

Proof. First, we claim that $\sqrt{\mathfrak{a}A_\mathfrak{p}}$ is the unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$ of the localisation. Indeed, suppose $\mathfrak{a}A_\mathfrak{p} \subseteq \mathfrak{n} \in \operatorname{Spec} A_\mathfrak{p}$. Contracting, we obtain $\mathfrak{a} \subseteq (\mathfrak{a}A_\mathfrak{p})^c \subseteq \mathfrak{n}^c \subseteq \mathfrak{p}$. But as $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{a}$, we must have $\mathfrak{n}^c = \mathfrak{p}$. Extending, $\mathfrak{n}^{ce} = \mathfrak{p}^e = \mathfrak{p}A_\mathfrak{p}$, but $\mathfrak{n}^{ce} = \mathfrak{n}$ as required. Hence, $\sqrt{\mathfrak{a}A_\mathfrak{p}}$ is the intersection of the primes containing it, which is just $\mathfrak{p}A_\mathfrak{p}$.

As the radical is maximal, the ideal $\mathfrak{a}A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$-primary. Note that $\mathfrak{a}A_\mathfrak{p} = \left(\frac{x_1}{1}, \dots, \frac{x_r}{1}\right)$, so by applying the dimension theorem,

$$\operatorname{ht}(\mathfrak{p}) = \dim A_\mathfrak{p} = \delta(A_\mathfrak{p}) \leq \delta(\mathfrak{a}A_\mathfrak{p}) \leq r$$

☐